Kochen and Specker's view on functional relations conflicts with the collapse postulate

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Introduction

A key ingredient of the Kochen-Specker theorem is the so-called functional composition principle, which asserts that hidden states must ascribe values to observables in a way that is consistent with all functional relations between them. This principle is motivated by the assumption that, like functions of observables in classical mechanics, a function $g(A)$ of an observable A in quantum theory is simply a logically possible observable derived from A, and that measuring $g(A)$ consists in measuring A and post-processing the resulting value via g. As Kochen and Specker put it, "the measurement of a function $g(A)$ of an observable A is independent of the theory considered — one merely writes $g(\boldsymbol{\alpha})$ for the value of $g(A)$ if $\boldsymbol{\alpha}$ is the measured value of A". Shortly speaking, we can say that, according to this view, $g(A)$ is "**a post-processing of** A **via** g".

Functional relations and the collapse postulate

If $g(A)$ represents an experimental post-processing of A via g, then the measurement event $(\beta, g(A))$, representing the experimental situation in which a measurement of $g(A)$ returns the outcome $\beta \in \sigma(g(A))$, has to be equivalent (in every possible way) to the measurement event $(g^{-1}(\beta), A)$, according to which A has been measured and some outcome lying in $g^{-1}(\beta)$ (unknown to the experimentalist) has been obtained. We see the following conditions as individually necessary and conjointly sufficient for the equivalence between $(\beta, g(A))$ and $(g^{-1}(\beta), A)$ in quantum theory:

I. $(\beta, g(A))$ and $(g^{-1}(\beta), A)$ are equally probable with respect to all states

II. $(\beta, g(A))$ and $(g^{-1}(\beta), A)$ update every state in precisely the same way.

As Kochen and Specker point out [1], it is easy to see that item I is satisfied by quantum theory. To analyse item II, we need to understand how the event $(g^{-1}(\beta), A)$ updates the state of the system. In our work, we consider the following definition:

Definition 1 (Collapse postulate including subjective events): Let A be any observable (selfadjoint operator) in a finite-dimensional Hilbert space H. When a measurement event (Δ, A) occurs, that is to say, when a measurement of A yields an outcome lying in $\Delta \subset \sigma(A)$ (unknown to the experimentalist), the state ρ of the system is updated to

$$\rho^A_\Delta \doteq \frac{1}{\mathrm{tr}(\rho E_\Delta)} \sum_{\alpha \in \Delta} E_\alpha \rho E_\alpha, \tag{1}$$

where E_α is the projection onto the subspace spanned by the eigenvalue α of A and $E_\Delta \equiv \sum_{\alpha \in \Delta} E_\alpha$.

It is easy to see that, according to this definition, $(\beta, g(A))$ and $(g^{-1}(\beta), A)$ do not necessarily update a state ρ in the same way, which leads us to the following theorem about quantum theory:

Theorem 1 The following statements about quantum theory cannot be simultaneously true.

(a) The standard collapse postulate (see, for instance, Ref. [2]) is correct.

(b) The collapse postulate including subjective events (definition 1) is correct.

(c) A function $g(A)$ of an observable A is the theoretical representation of an experimental post-processing of A via g

Discussion

In our work, we argue that the most reasonable way of avoiding theorem 1 consists in renouncing the standard collapse postulate. As we see it, the update must depend on a particular choice of "measurement basis" or "measurement context":

Definition 2 (context-dependent collapse) Let A be a selfadjoint operator in a n-dimensional Hilbert space H, and let $\mathfrak{B} \equiv \{E_i\}_{i=1}^n$ be a *measurement basis* for A, that is to say, $\mathfrak{B}$ is a set of rank-one pairwise orthogonal projections satisfying, for any $i \in \{1, \ldots, n\}$, $E_i A = \alpha_i E_i = A E_i$, where $\sigma(A) = \{\alpha_i : i = 1, \ldots, n\}$ is the spectrum of A. If a measurement of A in the basis $\mathfrak{B}$ yields an outcome α of A, the state ρ of the system is updated to

$$\rho^{(A,\mathfrak{B})}_\alpha \doteq \sum_{\substack{i=1 \\ \alpha_i = \alpha}}^{n} \frac{E_i \rho E_i}{\mathrm{tr}(\rho E_\alpha)}. \tag{2}$$

Based on this definition, we discuss the following points.

- There is more than one measurement basis (or measurement context) for an observable A if and only if A is **degenerate**, i.e., iff A has at least one degenerate eigenvalue (we say that A is nondegenerate otherwise). This is equivalent to saying that A can be written as a function $A = g(B) = h(C)$ of noncommuting observables B, C, which in turn is precisely the reason why noncontextual hidden variable models for quantum systems are ruled out by Kochen-Specker theorem [3, 1]. Hence, the dependence on contexts which follows from definition 2 is in agreement with the context dependence which arises from Kochen-Specker theorem
- Degenerate observables can always be seen as coarse-grainings of nondegenerate ones, which means that, if B is a degenerate observable, then there is a nondegenerate observable A and a (necessarily) non-injective function $g : \sigma(A) \to \sigma(B)$ such that $B = g(A)$. The distinction between degenerate and nondegenerate observables resembles the distinction between mixed and pure states
- The multiplicity of measurement bases for a degenerate observable is similar to the variety of convex decompositions of a mixed state, and the fact that a nondegenerate observable has a unique basis is comparable to the unique convex decomposition of a pure state. In Spekkens' contextuality [4], distinct convex combinations of a mixed state ρ are associated with distinct preparation procedures for ρ [4], and, as we argue in the paper, distinct measurement bases for a degenerate observable A are associated with distinct measurement procedures for A. Thus, the dependence on contexts that appears in definition 2 resembles Spekkens' notion of contextuality.
- With respect to the same measurement basis, the events $(\beta, g(A))$ and $(g^{-1}(\beta), A)$ are equivalent, i.e., they satisfy items I and II introduced above. Therefore, definition 2 allows us to avoid theorem 1 without rejecting Kochen and Specker's view on functional relations.

References

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