A causal modelling framework for classical and quantum cyclic causal structures

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Motivation

Connecting causal explanation to correlations is central to science and gives interesting insights of our theories. The theoretical existence of closed time-like curves (CTCs), curves that allow a particle to return to its starting point in space-time, as solutions in General Relativity seem to imply that time travel backwards in time is theoretically possible. The **description and characterisation of CTCs** from a causal point of view requires the introduction of **cyclic causal models**. These models do not only describe CTCs that may arise in exotic solutions of General Relativity, but also can be used to model ordinary feedback processes [1].

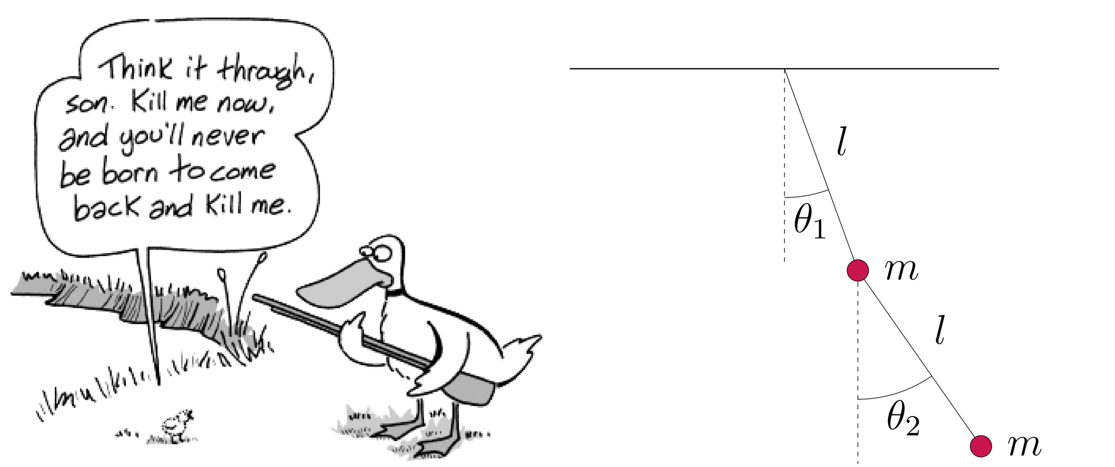


Figure 1: Grand-father paradox arising in CTCs and a double pendulum, physical process.

Classical causal models

Classical causal modelling [2] makes use of directed graphs to describe causal relations. Nodes of the graph represent variables, while edges represent direct causation expressed by a functional model $f_i(\mathsf{Pa}(X_i), A_i) = X_i$, where $\mathsf{Pa}(X_i) := \{X_j | X_j \to X_i\}$ and A_i are parentless variables.

Definition: The model is **uniquely solvable** if $\#\mathsf{sol}(\{a_i\}_i) = 1 \; \forall \{a_i\}_i$.

Definition: A probability distribution P is said to be **Markov relative** to a graph G if $P(x_1, \dots, x_n) = \prod_i P(x_i | \mathsf{pa}(x_i))$.

Classical causal modelling only provides probability distribution for cyclic classical casual structures that are uniquely solvable [1, 3]

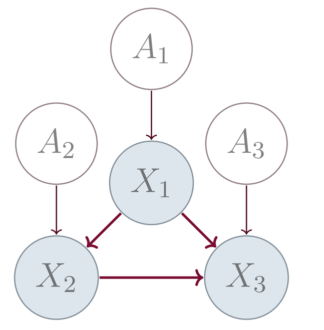


Figure 2: Example of classical acyclic causal model

Process matrix framework

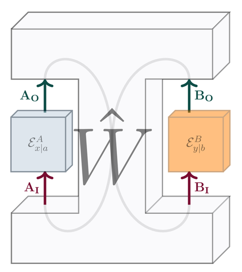


Figure 3: Example of process matrix

In the quantum case, a recent framework for **quantum cyclic causal models** has been proposed [4]. The nodes correspond to local laboratories of agents where they may perform quantum operations, while the edges denote quantum channels connecting different laboratories. This framework considers only cyclic causal models that remain non-paradoxical for all possible choices of local operations plugged in at the nodes and for them provides a probability distribution.

These are so-called **process matrices** [5] which are known to correspond to a linear subset of more general CTCs [6].

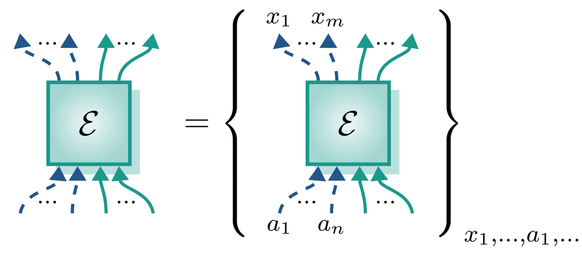
Aim

We formulate a diagram semantics that allows to model cyclic causal structures as acyclic ones with post-selection. This framework can consistently describe quantum cyclic causal models determining the existence or not of a **logically consistent solutions** and eventually provide a way to **evaluate probability distributions**.

This framework provides new insights on cyclic causal models and reproduces the known results as special cases.

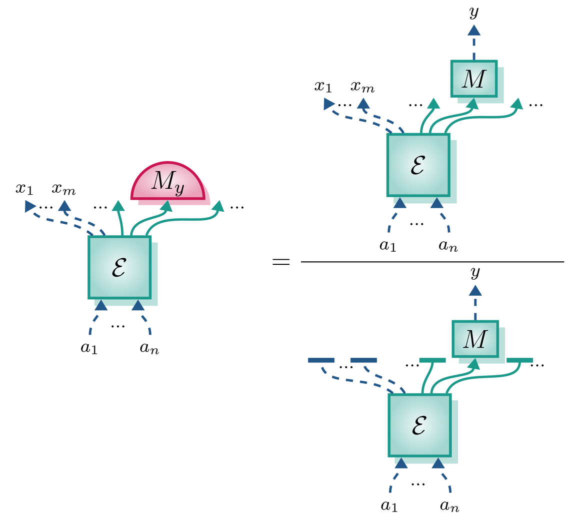
The framework

The building blocks of our notation are a collection of diagrams, where each dashed arrow carries a finite set and each solid arrow a Hilbert space. Each diagram represents a collection of completely positive linear maps from linear operators acting on the tensor product of input Hilbert spaces to linear operators acting on the tensor product of output Hilbert space, satisfying:



$$\forall a_1, \dots, a_n : \sum_{x_1, \dots, x_m} \mathcal{E} \text{ is trace-preserving.}$$

We can denote **post selection** on the outcome y of a channel M as a M_y box. Here the ground symbol represents marginalisation and the ground symbol performing the partial trace.



Causal model

A causal model on a directed graph $G = (V, E)$ is specified by the following items:

(i) a Hilbert space $\mathcal{H}(e)$ associated to each edge $e \in E$;

(ii) a finite set of settings $\mathcal{A}(v)$ and a finite set of outcomes $\mathcal{X}(v)$ associated to each vertex $v \in V$;

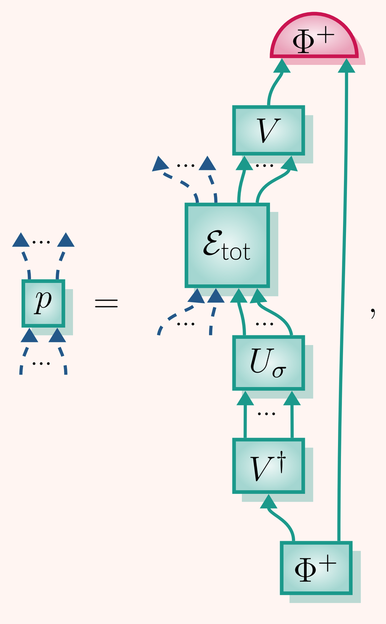
(iii) a completely positive linear map

$$\mathcal{E}^{(v)} : \mathcal{L}\left(\otimes_{e \in \mathrm{In}(v)} \mathcal{H}(e)\right) \to \mathcal{L}\left(\otimes_{e \in \mathrm{Out}(v)} \mathcal{H}(e)\right)$$

associated to each vertex $v \in V$, outcome $x \in \mathcal{X}(v)$ and setting $a \in \mathcal{A}(v)$, that satisfies the normalisation condition.

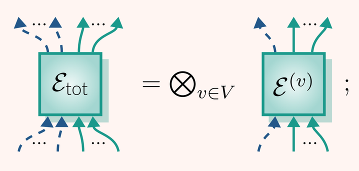
Probability distribution

Given a causal model on a directed graph $G = (V, E)$, we define:



where:

- $\mathcal{E}_{tot}$ is defined as:



- V is an isometry which reduces the set of output edges, where each edge carries a Hilbert space, to a single arrow carrying the tensor product of the Hilbert spaces;
- U_σ is a permutation of the edges which reorders the output edges in the order of input edges.

Explicitly:

$$P(x_1, \dots, x_n | a_1, \dots, a_n) = \frac{\langle \Phi^+ | (V \otimes \mathcal{I}_{B'})(\mathcal{E}_{tot} \otimes \mathcal{I}'_B) \left[(U_\sigma V^\dagger \otimes \mathcal{I}_{B'}) |\Phi^+\rangle\langle\Phi^+| \right] |\Phi^+\rangle}{\mathcal{N}}$$

Results

- The framework can be specialised to describe **classical functional causal models**. The probability distribution can be expressed in terms of the distribution of an acyclic model where the number of variables are doubled (P_{se}):

$$P(x_1, \dots, x_n | a_1, \dots, a_n) = \frac{\sum_{\{y_j\}_j} P_{\mathsf{se}}(x_1, \dots, x_n | y_1, \dots, y_n; a_1, \dots, a_n) \prod_i \delta_{y_i, x_i}}{\sum_{\{x_j\}_j, \{y_j\}_j} P_{\mathsf{se}}(x_1, \dots, x_n | y_1, \dots, y_n; a_1, \dots, a_n) \prod_i \delta_{y_i, x_i}} \quad (1)$$

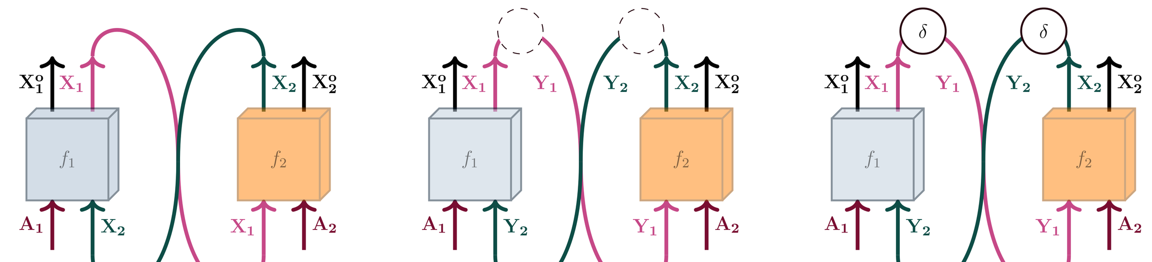


Figure 4: Interpretation of the classical probability distribution with a two-node example.

- In the classical case, it provides a characterisation of models where the Markov condition holds:

Definition: A model is **averagely unique solvable** (AUS) if $\langle \#\mathsf{sol}(A_i) \rangle_P = \sum_{\{a_i\}_i} P(a_i) \#\mathsf{sol}(a_i) = 1$.

Given a causal model, eq (1) admits a Markov factorisation $\iff$ the model is AUS.

- We discuss **composability** with two different methods of describing input variables.
- We prove explicitly the equivalence between the **post-selected teleportation protocol and loop composition** defined in the causal box framework [7].
- We formulate a generalisation of the **d-separation theorem** [2, 8] to quantum cyclic causal models.

Outlook

Our framework provides a method to uniquely determine probability distributions of **arbitrary classical and quantum cyclic causal models**, generalising previously known approaches for quantum cyclic causal models [4, 9]. It connects quantum cyclic causal models to quantum acyclic causal models with post-selection allowing to directly generalise results from the acyclic case to the cyclic one through this correspondence. It is formulated rather operationally in terms of composition of operations and post-selection, and has the scope to be generalised in a more theory independent manner to **post-quantum operational theories** (i.e. to any physical theory that has an analogue of post-selected teleportation).

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References

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