No state-independent contextuality from state-dependent contextual MBQC in odd prime dimension

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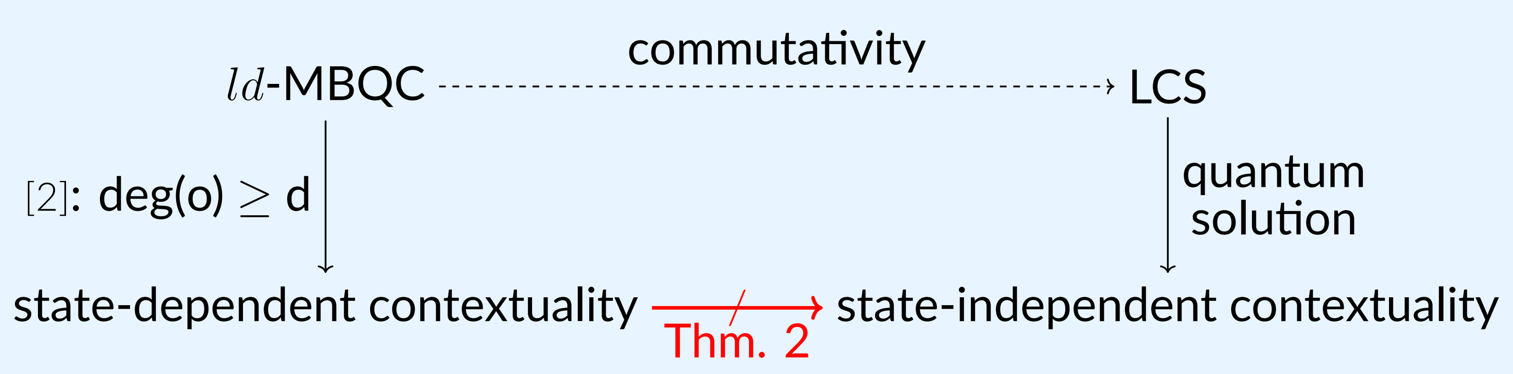
Summary



Linear constraint systems (LCS)

Background

A linear constraint system (LCS) over $\mathbb{Z}_d$ is a set of linear equations '$Ax = b \mod d$', where $A \in M_{m\times n}(\mathbb{Z}_d)$, $b \in \mathbb{Z}_d^m$. A **classical solution** is a vector $x = (x_k)_{k=1}^n \in \mathbb{Z}_d^n$ such that $Ax = b \mod d$. A **quantum solution** is a set of unitary operators $(x_k)_{k=1}^n$, $x_k \in U(\mathbb{C}^d)$ such that

- **d-torsion:** $x_k^d = \mathbb{1}$
- **commutativity:** $[x_k, x_{k'}] = x_k x_{k'} x_k^{-1} x_{k'}^{-1} = \mathbb{1}$ whenever $A_{jk} \neq 0 \neq A_{jk'}$ for some $j \in [m] := \{1, \cdots, m\}$
- **constraint satisfaction:** $\prod_{k=1}^n x_k^{A_{jk}} = \omega^{b_j}\mathbb{1}$ for all $j \in [m]$ and $\omega = e^{\frac{2\pi i}{d}}$.

Measurement-based quantum computation (MBQC)

Background

A deterministic, non-adaptive ld-MBQC is given by the following data:

- **resource state:** $|\psi\rangle \in (\mathbb{C}^d)^N$ (e.g. the N-qudit GHZ-state $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{q=0}^{d-1}|q\rangle^{\otimes N}$)
- **local measurement operators:** $M_k \in U(\mathbb{C}^d)$ with eigenvalues d-th roots of unity $Z := \{\omega^a \mid a \in \mathbb{Z}_d\}$; consequently, $M_k^d = \mathbb{1}$ for all $k \in [N]$ (d-torsion)
- **measurement settings:** $M_k = M_k(l_k)$, where $l_k : \mathbb{Z}_d^n \to \mathbb{Z}_d$ is a $\mathbb{Z}_d$-linear function of the input vector $\mathbf{i} \in \mathbb{Z}_d^n$
- **output:** The output $\mathbb{Z}_d \ni o = \sum_{k=1}^N m_k \mod d$ is the sum of local measurement outcomes. For deterministic (non-adaptive, ld-) MBQC, the output defines a function $o : \mathbb{Z}_d^n \to \mathbb{Z}_d$.

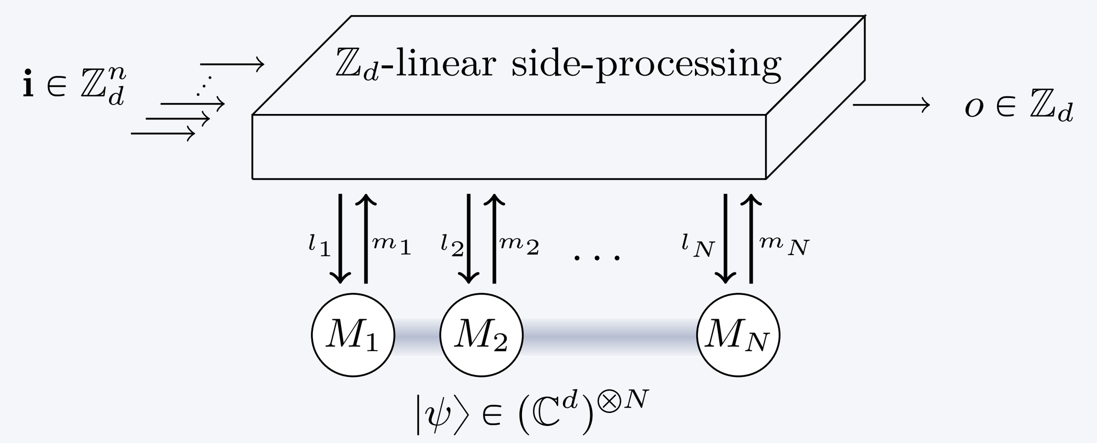


Figure 1. The schematic setup of ld-MBQC [2].

LCS from MBQC

Background

We associate to any deterministic (non-adaptive ld-) MBQC a LCS '$Ly = o \mod d$', where

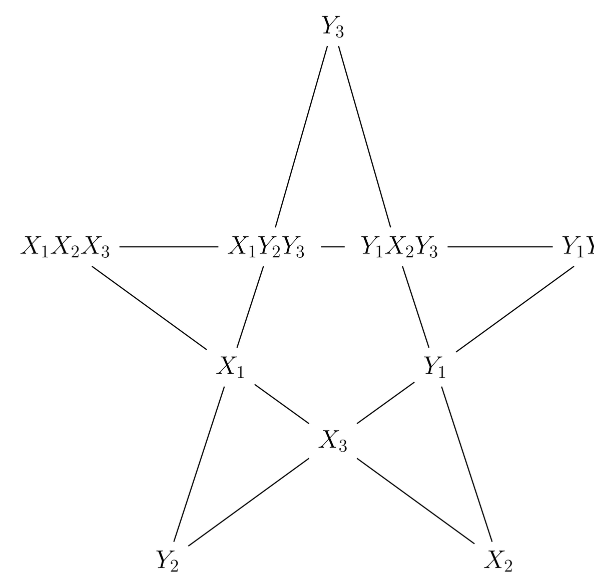
$$L = (L_{\mathbf{i},k})_{k=1,\mathbf{i}\in\mathbb{Z}_d^n}^N = (l_1(\mathbf{i}), \cdots, l_N(\mathbf{i})) \in M_{d^n\times N}(\mathbb{Z}_2), \quad N = d^n, \mathbf{i} \in \mathbb{Z}_d^n$$

is the matrix of measurement settings and $o : \mathbb{Z}_d^n \to \mathbb{Z}_d$, equivalently $o \in \mathbb{Z}_d^N$, is the output of the MBQC. **Note:** hidden variable model of MBQC $\longleftrightarrow$ classical solution of associated LCS

Does a contextual MBQC give rise to a quantum solution of its associated LCS?

Contextuality in MBQC

Example



(a) Mermin's star.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$M_k(l_k) = S_k^{l_k}X_k = \begin{cases} X_k & \text{for } l_k = 0 \\ Y_k & \text{for } l_k = 1 \end{cases}, \qquad S_k := \mathrm{diag}(-i, i)$$

$$L_{\mathbf{i}k} := (l_1(\mathbf{i}), l_2(\mathbf{i}), l_3(\mathbf{i})) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad o(\mathbf{i}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) MBQC and associated LCS.

Figure 2. Qubit Pauli operators in Mermin's star [4] (tensor products and identity operators omitted).

Note: $l_1(\mathbf{i}) = i_1$, $l_2(\mathbf{i}) = i_2$, $l_3(\mathbf{i}) = i_1 + i_2 \mod 2$ are $\mathbb{Z}_2$-linear, yet $o(\mathbf{i}) = i_1 i_2 + i_1 + i_2 \mod 2$ is the nonlinear (and universal for classical computation) OR-gate [1]

- **nonlinearity of $o : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is a witness of contextuality in deterministic $l2$-MBQC** [6]
- **contextuality boosts the classical computer beyond $\mathbb{Z}_2$-linear side-processing**

The above example generalises to arbitrary prime dimension d. We write $d = p$ for p prime.

Theorem 1

Let $|\psi\rangle = \frac{1}{\sqrt{p}}\sum_{q=0}^{p-1}|q\rangle^{\otimes N}$, for $N = 3$ and p prime. The lp-MBQC with local measurements

$$M_k(l_k) := S_{\xi_k}^{l_k}X_k, \quad \xi_k(q_k) = \exp\left(l_k\frac{2\pi i}{p^2}(1 + p(q_k - 1)^{d-1})\right) \qquad \forall q_k, l_k \in \mathbb{Z}_p ,$$

set by $\mathbb{Z}_p$-linear functions $l_1(\mathbf{i}) = i_1$, $l_2(\mathbf{i}) = i_2$, and $l_3(\mathbf{i}) = -i_1 - i_2 \mod p$ on the input $\mathbf{i} = (i_1, i_2)^T \in \mathbb{Z}_p^2$ is contextual, i.e., it admits no noncontextual hidden variable model.

Remarks

- more generally, **local measurement operators in Eq. (1) are universal for contextual MQBC:** any output function $o : \mathbb{Z}_d^n \to \mathbb{Z}_d$ can be computed from these operators [3]
- the operators in Eq. (1) are **generally not Pauli operators**

Pauli group, Heisenberg-Weyl group, and (diagonal) Clifford hierarchy

Background

We denote by $\mathcal{P}_d^{\otimes N}$ the N-qudit **Pauli group**, where $\mathcal{P}_d^1 = \langle X, Z, \tau\mathbb{1}\rangle$ is generated by the generalised Pauli-X and -Z operators defined by (their action on the computational basis $\{|q\rangle\}_{q=0}^{d-1}$),

$$X|q\rangle = |q + 1 \mod d\rangle \qquad Z|q\rangle = \omega^q|q\rangle \quad \forall q \in \mathbb{Z}_d ,$$

and $\tau = \omega = e^{\frac{2\pi i}{d}}$ for d odd, whereas $\tau = \sqrt{\omega}$ for d even. For d odd, $\mathcal{P}_d^1 \cong \mathsf{H}(\mathbb{Z}_d)$, where $\mathsf{H}(\mathbb{Z}_d) = \langle \omega^{-ab}Z^aX^b \mid a, b \in \mathbb{Z}_d\rangle$ is the **discrete Heisenberg-Weyl group**.

The **diagonal Clifford hierarchy** is defined recursively from $\mathcal{C}_1(d) := \{S_\xi \mid \xi(q) = e^{i\phi}\omega^{\beta q}, \phi \in [0, 2\pi), \beta \in \mathbb{Z}_d\}$, and $\mathcal{C}_{k+1}(d) := \{S_\xi = \mathrm{diag}(\xi(0), \cdots, \xi(d-1)) \in U(d) \mid XS_\xi X^{-1} \subset \mathcal{C}_k(d)\}]$.

We also define $\mathcal{SC}_k(d) := \mathcal{C}_k(d) \cap SU(d)$. Note that $\mathcal{SC}_1(d) = \{S_\xi \mid \xi(q) = \omega^{\alpha+\beta q}, \alpha, \beta \in \mathbb{Z}_d\}$.

Group of measurement operators

The operators in the Heisenberg-Weyl group $\mathsf{H}(\mathbb{Z}_d)$ generalise to (local measurement) operators

$$M(\xi, b) := S_\xi X^b \qquad S_\xi|q\rangle = \xi(q)|q\rangle \quad \forall q \in \mathbb{Z}_d , \tag{1}$$

where $\xi : \mathbb{Z}_d \to U(1)$. The d-torsion condition for $M(\xi, b \neq 0)$ becomes $\prod_{q=0}^{d-1}\xi(q) = 1$, equivalently $S_\xi \in T$ for a maximal torus of the special unitary group $SU(d)$,

$$T = T(SU(d)) = \{S_\xi = \mathrm{diag}(\xi(0), \cdots, \xi(d-1)) \in SU(d) \mid \xi : \mathbb{Z}_d \to U(1)\} .$$

Definition: *Let $Q \subset \mathcal{SC}_k(d) \subset T$ for some $k \in \mathbb{N}$, closed under translations $t : \mathbb{Z}_d \to \mathrm{Aut}(Q)$, $t.S_\xi := XS_\xi X^{-1}$. We define $K_Q(d) := \langle M(\xi, b) := S_\xi X^b \mid S_\xi \in Q, b \in \mathbb{Z}_d\rangle \subset SU(d)$. We write $K_Q^{\otimes N}(d) := \otimes_{k=1}^N(K_Q(d))_k$ for the N-fold tensor product of $K_Q(d)$.*

Reduction to Heisenberg-Weyl group

Recall: measurements operators in MBQC only commute on the common eigenstate $|\psi\rangle$!

problem: When do elements in $K_Q^{\otimes N}(p)$ commute?

Lemma: *Let $M, M' \in K_Q(p)$ for p prime. Then $[M, M'] = \omega^c$, $c \in \mathbb{Z}_p$ if and only if either*

(i) *$M, M' \in Q$, or*
(ii) *either $M \in Q(\mathsf{H}(\mathbb{Z}_p)) = \mathcal{SC}_1(p)$ or $M' \in Q(\mathsf{H}(\mathbb{Z}_p)) = \mathcal{SC}_1(p)$, or*
(iii) *$M' = (WM)^y$ for $W \in Q(\mathsf{H}(\mathbb{Z}_p)) = \mathcal{SC}_1(p)$ and $y \in \mathbb{Z}_p$.*

The close relationship between abelian subgroups in $K_Q^{\otimes}(p)$ and $\mathsf{H}^{\otimes}(\mathbb{Z}_p)$ gives rise to a map $\phi : K_Q^{\otimes N}(p)_p \to \mathsf{H}^{\otimes N}(\mathbb{Z}_p)$—a homomorphism in abelian subgroups of order p in $K_Q^{\otimes N}(p)$.

Theorem 2

Let $Ly = o \mod p$ be a LCS over $\mathbb{Z}_p$ for p odd prime. Then the LCS admits a quantum solution in $K_Q^{\otimes N}(p) \subset SU(p)$ with $Q \subset \mathcal{SC}_k(p)$ for some $k \in \mathbb{N}$ if and only if it admits a classical solution.

Sketch of proof: ϕ preserves the constraints of the LCS

(i) **d-torsion:** $\phi^p(M) = \phi(M^p) = \phi(1) = 1$. (Every operator $P \in \mathsf{H}^{\otimes N}(\mathbb{Z}_p)$ has order p.)
(ii) **commutativity:** $\phi(MM') = \phi(M)\phi(M')$ whenever $[M, M'] = 1$.
(iii) **constraint satisfaction:** let $\{M_k\}_{k\in J}$, $M_k \in K_Q^{\otimes N}(p)$ be a set of pairwise commuting operators and $\omega^{b_J} = \prod_{k\in J} M_k$, then $\phi(\prod_{k\in J} M_k) = \prod_{k\in J} M_k = \omega^{b_J} = \phi(\omega^{b_J})$.

Hence, ϕ maps (classical/quantum) solutions of the LCS $Ly = o \mod p$ in $K_Q^{\otimes N}(p)$ to (classical/quantum) solutions in $\mathsf{H}^{\otimes N}(\mathbb{Z}_p)$; by [5], the latter are classical.

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