

Building qutrit diagonal gates from phase gadgets

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Phase gadgets have proved to be an indispensable tool for reasoning about ZX-diagrams, being used in optimisation and simulation of quantum circuits and the theory of measurement-based quantum computation. In this paper we study phase gadgets for qutrits. We present the *flexsymmetric* variant of the original qutrit ZX-calculus, which allows for rewriting that is closer in spirit to the original (qubit) ZX-calculus. In this calculus phase gadgets look as you would expect, but there are non-trivial differences in their properties. We devise new qutrit-specific tricks to extend the graphical Fourier theory of qubits, resulting in a translation between the ‘additive’ phase gadgets and a ‘multiplicative’ counterpart we dub *phase multipliers*.

This enables us to generalise the qubit notion of multiple-control to qutrits in two ways. The first type is controlling on a single tritstring, while the second type applies the gate a number of times equal to the tritwise multiplication modulo 3 of the control qutrits. We show how both types of control can be implemented for any qutrit Z or X phase gate, ancilla-free, and using only Clifford and phase gates. The first requires a polynomial number of gates and exponentially small phases, while the second requires an exponential number of gates, but constant sized phases. This is interesting, because such a construction is not possible in the qubit setting.

As an application of these results we find a construction for *emulating* arbitrary qubit diagonal unitaries, and specifically find an ancilla-free emulation for the qubit CCZ gate that only requires three single-qutrit non-Clifford gates — provably lower than the four T gates needed for qubits with ancilla.

1 Introduction

Most quantum computing theory developed thus far has focussed on qubits — two-level quantum systems. However, there has been a recent surge of interest in studying the more general case of d -level quantum systems, called *qudits*. This has led to applications of qudits for quantum algorithms [52], improving magic state distillation noise thresholds [12], and communication noise resilience [20]. Qudits have been experimentally demonstrated on quantum processors based on ion traps [46] and superconducting devices [8, 55, 57, 34].

The specific case of *qutrits*, where $d = 3$, has been used to improve qubit readout [40], but most notably, qutrits have been used to study *emulation*: where qubit computation is emulated inside a subspace of the qudits to enable more resource-efficient gate implementations. In contrast, qubits cannot simulate qudit (where $d > 2$) computation efficiently [11].

Much work on qutrits and emulation has focussed on *classical* functions: those that come from a map of classical trits. For instance, using qutrits we can build logarithmic-depth Toffolis [27, 41] and binary AND gates on superconducting qutrits [16]. This leaves open the question of whether there are any advantages to emulation by studying ‘truly’ quantum gates such as diagonal unitaries. For qubits a useful tool for understanding diagonal unitaries has been the concept of a *phase gadget* [38]. This is a type of symmetric multi-qubit interaction that occurs naturally in many hardware architectures [43, 42, 47], and serves as a good basis for optimising quantum circuits [18, 19, 7, 6, 54, 5]. Any diagonal qubit unitary can be expressed as a product of phase gadgets by writing the unitary as a *phase polynomial* [1, 31].

In this paper we study the generalisation of phase gadgets to the qutrit setting. We do this by adapting the qutrit ZX-calculus of Refs. [28, 50] and transforming it into a *flexsymmetric* calculus [15] where the spiders have more desirable symmetry properties. We find this calculus has a simple set of rules for the Clifford fragment. We define phase gadgets analogously to the qubit case, meaning that as diagrams they look nearly identical. There are however significant differences between the qubit and qutrit gadgets. We will show that we can nevertheless use qutrit phase gadgets to construct some useful qutrit diagonal unitaries, such as controlled phase gates, and a type of gate we dub a *phase multiplier*. This last one is possible by generalising the formula that leads to the *graphical Fourier theory* for qubit diagonal unitaries [39].

As an application of our results we show how we can emulate an arbitrary qubit diagonal unitary using qutrit phase gadgets. This leads us to a construction of the emulated qubit CCZ gate that requires only three non-Clifford qutrit *R gates* [26]. This is surprising because using just qubits, we would require at least four *T gates* to implement the CCZ [35].

We start the paper by reviewing the basics of qutrit quantum computation in Section 2. Then we introduce the flexsymmetric qutrit ZX-calculus in Section 3. Diagonal qutrit unitaries, phase gadgets, controlled phase gates, and phase multipliers are studied in Section 4. We show how to use these to emulate diagonal qubit unitaries in Section 5 and end with some discussion on future work in Section 6.

2 Qutrit quantum computation

A qubit is a two-dimensional Hilbert space. Similarly, a qutrit is a three-dimensional Hilbert space. We will write $|0\rangle$, $|1\rangle$, and $|2\rangle$ for the standard computational basis states of a qutrit. Any normalised qutrit state can then be written as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$ where $\alpha, \beta, \gamma \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

Several concepts for qubits extend to qutrits, or more generally to *qudits*, which are d -dimensional quantum systems. In particular, the concept of Pauli's and Cliffords. For a d -dimensional qudit, we define the respective Pauli X and Z gates as

$$X|k\rangle = |k+1\rangle \quad Z|k\rangle = \omega^k|k\rangle \quad (1)$$

where $\omega := e^{2\pi i/d}$ is such that $\omega^d = 1$, and the addition $|k+1\rangle$ is taken modulo d [30, 36]. We define the *Pauli group* as the set of unitaries generated by tensor products of the X and Z gate. For qubits this X gate is just the NOT gate, while $Z = \text{diag}(1, -1)$. In this paper we will work solely with qutrits, so we take ω to always be equal to $e^{2\pi i/3}$. Note that $\omega^{-1} = \omega^2 = \bar{\omega}$ where \bar{z} denotes the complex conjugate of z .

For a qubit there is only one non-trivial permutation of the standard basis states, implemented by the X gate. For qutrits there are five non-trivial permutations of the basis states. By analogy we will call them all ternary X gates. These gates are X_{+1} , X_{-1} , X_{01} , X_{12} , and X_{02} . $X_{\pm 1}$ sends $|t\rangle$ to $|(t \pm 1) \bmod 3\rangle$ for $t \in \{0, 1, 2\}$; X_{01} is just the qubit X gate which is the identity when the input is $|2\rangle$; X_{12} sends $|1\rangle$ to $|2\rangle$ and $|2\rangle$ to $|1\rangle$, and likewise for X_{02} . Note that the qutrit Pauli X gate is the X_{+1} gate, while $X^\dagger = X_{-1} = X^2$.

Another concept that translates to qutrits (or more generally qudits) is that of Clifford unitaries.

Definition 2.1. Let U be a qudit unitary acting on n qudits. We say it is *Clifford* when every Pauli is mapped to another Pauli under conjugation by U . I.e. if UPU^\dagger is in the Pauli group for any Pauli P .

The set of n -qudit Cliffords forms a group under composition. For qubits, this group is generated by the S , Hadamard and CX gates. The same is true for qutrits, for the right generalisation of these gates¹.

¹The gate definitions for various qudit Cliffords may vary across the literature up to a global phase. Indeed, by Definition 2.1, whether a gate is Clifford is invariant under changes in global phase

Definition 2.2. The qutrit S gate is $S := \text{diag}(1, 1, \omega)$. I.e. it multiplies the $|2\rangle$ state by the phase ω .

For qubits, the Hadamard gate interchanges the Z eigenbasis $\{|0\rangle, |1\rangle$ and the X eigenbasis consisting of the states $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The same holds for the qutrit Hadamard. In this case the X basis consists of the following states:

$$|+\rangle := \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \quad |\omega\rangle := \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \bar{\omega}|2\rangle) \quad |\bar{\omega}\rangle := \frac{1}{\sqrt{3}}(|0\rangle + \bar{\omega}|1\rangle + \omega|2\rangle)$$

Definition 2.3. The qutrit Hadamard gate H is the unitary mapping $|0\rangle \mapsto |+\rangle$, $|1\rangle \mapsto |\omega\rangle$ and $|2\rangle \mapsto |\bar{\omega}\rangle$.

$$H := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix} \quad (2)$$

Note that, unlike the qubit Hadamard, the qutrit Hadamard is *not* self-inverse. Instead we have $H^2 = X_{12}$ so that $H^4 = \mathbb{I}$. This means that $H^\dagger = H^3$.

The final Clifford gate we need is the qutrit CX.

Definition 2.4. The qutrit CX is defined such that $\text{CX}|i, j\rangle = |i, (i+j) \bmod 3\rangle$, where $i, j \in \{0, 1, 2\}$.

Any qutrit Clifford unitary can be written as a composition of S , H and CX gates (up to global phase). Clifford gates are efficiently classically simulable, so we need to add another gate to get an (approximately) universal gate set for quantum computing [30]. This will be a phase gate.

Definition 2.5. We write $Z(a, b)$ for the *phase gate* that acts as $Z(a, b)|0\rangle = |0\rangle$, $Z(a, b)|1\rangle = \omega^a|1\rangle$ and $Z(a, b)|2\rangle = \omega^b|2\rangle$ where we take $a, b \in \mathbb{R}$.

We define $Z(a, b)$ in this way, taking a and b to correspond to powers of ω , because then $Z(a, b)$ is Clifford iff a and b are both integers, so that we can easily see from the parameters whether the gate is Clifford or not. The group of $Z(a, b)$ phase gates constitutes the group of diagonal single-qutrit unitaries modded out by a global phase. Composition of these gates is given by $Z(a, b) \cdot Z(c, d) = Z(a+c, b+d)$. Note that $S = Z(0, 1)$. This brings us to the definition of the qutrit T gate.

Definition 2.6. The qutrit T gate is defined as $T := Z(\frac{1}{3}, -\frac{1}{3}) = \text{diag}(1, \omega^{\frac{1}{3}}, \omega^{-\frac{1}{3}})$ [44, 13, 36].

Like the qubit T gate, the qutrit T gate belongs to the third level of the Clifford hierarchy, can be injected into a circuit using magic states, and its magic states can be distilled by magic state distillation. This means that we can fault-tolerantly implement this qutrit T gate on many types of quantum error correcting codes. Also as for qubits, the qutrit Clifford+ T gate set is approximately universal, meaning that we can approximate any qutrit unitary using just Clifford gates and the T gate [23, Theorem 1].

There is another useful single-qutrit non-Clifford gate.

Definition 2.7. The qutrit *reflection* gate is defined as $R := Z(0, 3/2) = \text{diag}(1, 1, -1)$.

Like the T gate, the R gate can be added to the Clifford gate set to attain universality [30], as explicitly proved in Ref. [23, Theorem 2]. It can be exactly synthesized fault-tolerantly in three known ways: magic state distillation followed by repeat-until-success injection [2], braiding and topological measurement of weakly-integral non-Abelian anyons [22, 23] followed by repeat-until-success injection [2], or unitarily in qutrit Clifford+ T [26].

We see that in this definition the inputs and outputs are treated on equal footing. In order to prevent confusion with earlier work, we will denote this new X-spider in pink, instead of in red².

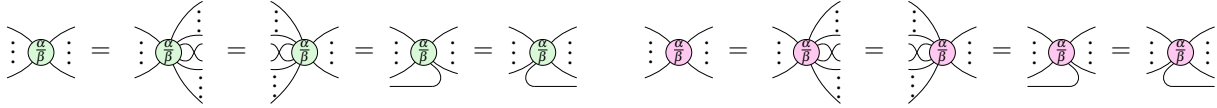
Let's now give the full definition of the spiders. We define the Z-spider as

$$\begin{array}{c} \vdots \\ \vdots \\ \textcircled{\frac{\alpha}{\beta}} \\ \vdots \\ \vdots \end{array} = |0 \cdots 0\rangle \langle 0 \cdots 0| + \omega^\alpha |1 \cdots 1\rangle \langle 1 \cdots 1| + \omega^\beta |2 \cdots 2\rangle \langle 2 \cdots 2|$$

Here we have two phase angles α and β , as opposed to just the one angle in qubit ZX. In general, for a d -dimensional spider, you will need to specify $d - 1$ phases. In particular, when written in a spider $\frac{\alpha}{\beta}$ should be interpreted as two different phases and not as the fraction α/β . Note that we define the phase angles as ω^α and ω^β so that these correspond to the complex phases $e^{i2\pi/3\alpha}$ and $e^{i2\pi/3\beta}$. This means that when α and β are integers, that the spiders correspond to the Clifford fragment of the calculus. We define the X-spider similarly, but with respect to the X-basis:

$$\begin{array}{c} \vdots \\ \vdots \\ \textcircled{\frac{\alpha}{\beta}} \\ \vdots \\ \vdots \end{array} = |+\cdots+\rangle \langle +\cdots+| + \omega^\alpha |\bar{\omega} \cdots \bar{\omega}\rangle \langle \omega \cdots \omega| + \omega^\beta |\omega \cdots \omega\rangle \langle \bar{\omega} \cdots \bar{\omega}|$$

This requires some explanation, because this does not look symmetric in the inputs and outputs. However, note that $\langle \omega| = (|\omega\rangle)^\dagger = (|0\rangle + \omega|1\rangle + \omega^2|2\rangle)^\dagger = \langle 0| + \bar{\omega}\langle 1| + \omega\langle 2|$. Hence, if we take the transpose of $|\omega\rangle$ we actually get $\langle \bar{\omega}|$. It is straightforward to check that with $\alpha = \beta = 0$ we get back Eq. (4). These definitions of the Z-spider and X-spider satisfy the symmetry properties we want, namely:



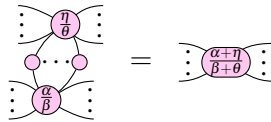
These symmetries mean our spiders are *flexsymmetric*, as defined by Carette [15], and as a result we may treat our ZX-diagrams as undirected graphs with the spiders as vertices. Note that here the cups and caps are defined with respect to the Z basis: $\subset = |00\rangle + |11\rangle + |22\rangle$. As usual, our calculus also formally has generators for the identity wire and the swap.

It will be useful to introduce an additional graphical generator for the Hadamard:

$$\boxed{\text{---}} = |+\rangle \langle 0| + |\omega\rangle \langle 1| + |\bar{\omega}\rangle \langle 2| = |0\rangle \langle +| + |1\rangle \langle \bar{\omega}| + |2\rangle \langle \omega| \tag{5}$$

We write the Hadamard as a slanted box, because it is self-transpose, but not self-adjoint, and so should be denoted in a way that is symmetric under a rotation, but not a reflection.

Our redefinition of the X-spider comes at a ‘cost’. Namely, the 1-input, 1-output X-spider is no longer the identity: $\text{---} \textcircled{\frac{\alpha}{\beta}} \text{---} = |0\rangle \langle 0| + |2\rangle \langle 1| + |1\rangle \langle 2| = |+\rangle \langle +| + |\bar{\omega}\rangle \langle \omega| + |\omega\rangle \langle \bar{\omega}|$. This map is implementing $|x\rangle \mapsto |-x\rangle$ where $-x$ is taken modulo 3, and is equal to X_{12} . Additionally, the X-spider is not really a spider any more in the sense that it doesn't satisfy the standard spider-fusion equation. Instead it satisfies the ‘harvestman equation’ [15] that also holds for for instance the W-spider [32] and H-box [4]:



In Figure 1, we present a full set of rewrite rules for this qutrit ZX-calculus. We have accounted for the global phase for each rule here as a complex number, as those will be relevant to us. Note however that the rewrite rules are not scalar-accurate as we are ignoring factors of $\sqrt{3}$.

²We have checked the accessibility of this color scheme; in fact, a red-green colorblind person greatly preferred this pink to the default ZX red.

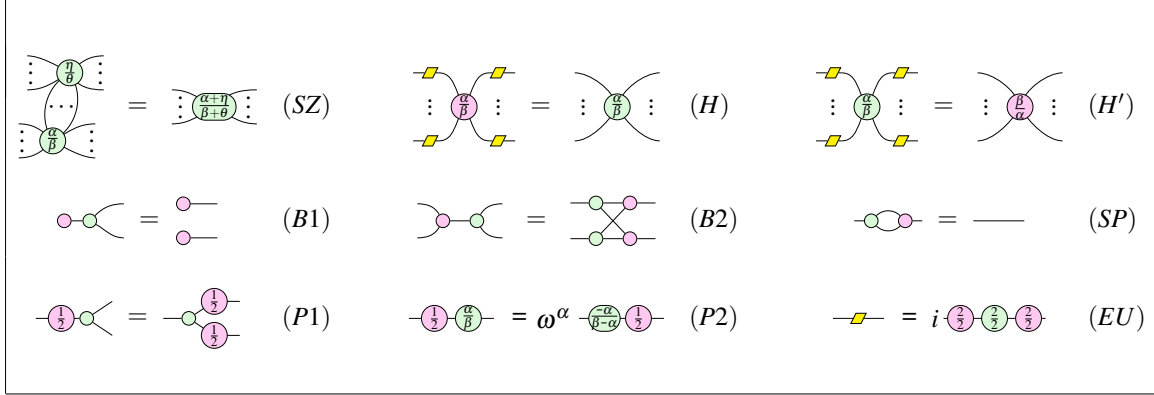


Figure 1: Rules for the flexsymmetric qutrit ZX-calculus. These hold for all $\alpha, \beta, \eta, \theta \in \mathbb{R}$, and for any permutation of the input and output wires. Additional useful derived rules are presented in Figure 2. The letters stand respectively for (S)pider-Z, (H)adamard, (B)ialgebra, (SP)ecial, (P)auli, and (EU)ler decomposition.

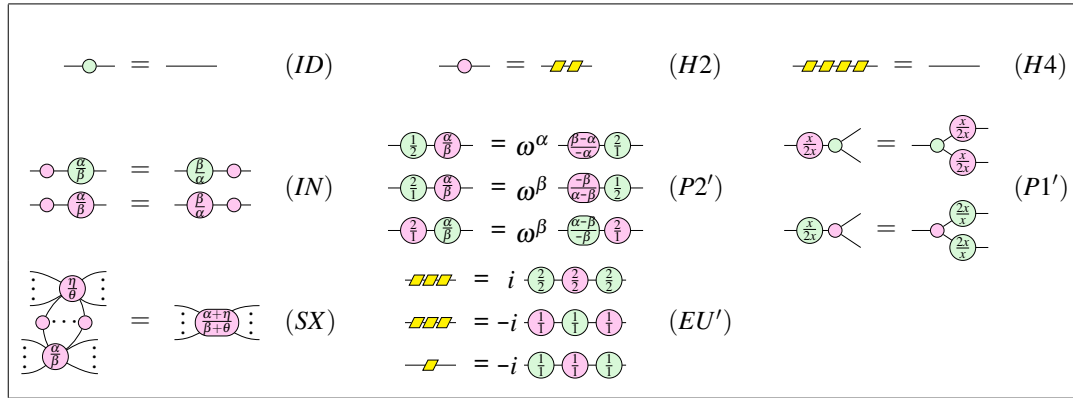


Figure 2: These rules are derivable from the rules of Figure 1 for any $\alpha, \beta, \eta, \theta \in \mathbb{R}$ and $x \in \{0, 1, 2\}$. The new letters stand respectively for (ID)entity, (IN)vert and (S)pider-X.

Using these rules, other useful qutrit ZX-calculus rewrite rules may be derived. In particular, we can use these rules to prove the derived rules presented in Figure 2. As these rules are (a slight variation) on the non-flexsymmetric qutrit rules of Ref. [50], our calculus is also complete for the qutrit Clifford fragment (when ignoring non-zero scalars). The proofs of the derived rules of Figure 2 are given in Appendix A.2. We show in Appendix A.1 that most rules of Figure 1 are necessary (i.e. not derivable from the others).

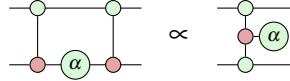
We see in these rules that there is a special role for phases of the form $\frac{x}{2x}$ where $x \in \{0, 1, 2\}$. This is because $\frac{x}{2x} \propto |x\rangle$ and $-\frac{x}{2x} = Z^x$. These relations can be derived by using the identity $1 + \omega + \bar{\omega} = 0$ together with $\omega^2 = \omega^{-1} = \bar{\omega}$. In general we will see a lot of $-\frac{\alpha}{2\alpha}$ phases because they implement the $|x\rangle \mapsto \omega^{\alpha x} |x\rangle$ phase gate. Additionally, note that the (P2) rule on the qubit subspace is exactly the familiar qubit ZX rule $-\pi \circ \alpha = e^{i\alpha} -\alpha \circ \pi$, since the red π is the qubit Pauli X while the pink $\frac{1}{2}$ phase is the qutrit Pauli X.

4 Diagonal qutrit gates

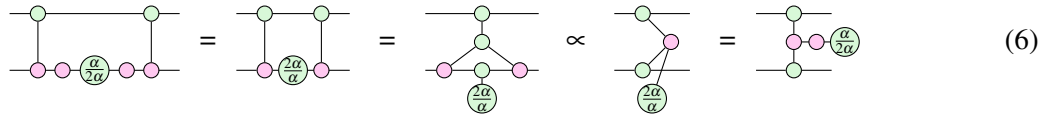
4.1 Phase gadgets

For qubits the concept of a *phase gadget* has proven very useful. There's several different ways we can define a qubit phase gadget. One way is to consider it as the diagonal gate $|x, y\rangle \mapsto e^{i\alpha(x \oplus y)} |x, y\rangle$ (for simplicity we are only considering a two-qubit phase gadget). This applies a phase of $e^{i\alpha}$ when $x \oplus y = 1$. Here \oplus is the XOR operation, which is the addition on \mathbb{Z}_2 . This suggests that we should define the qutrit phase gadget as $|x, y\rangle \mapsto e^{i\alpha(x+y)} |x, y\rangle$ where now we take $x + y$ to be modulo 3.

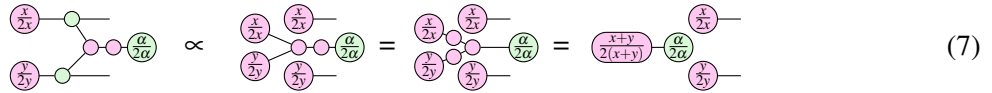
We could also define a phase gadget by its circuit realisation or diagrammatic representation. For qubits [38]:



We claim the qutrit variant of this construction is given by the following circuit which can be simplified to a similar diagrammatic representation:

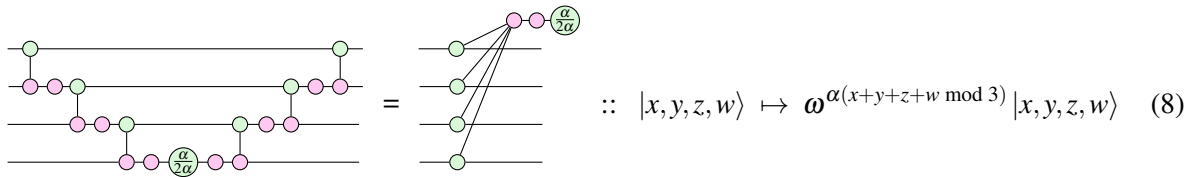


Indeed, inputting $|x, y\rangle$ into this diagram allows us to calculate its action:



This ‘floating scalar’ expression evaluates to $\sqrt{3}\omega^{\alpha(x+y \bmod 3)}$, so that this diagram indeed implements the operation we want, and we see that these three ways to define a qubit phase gadget—via the action, via the circuit, or via the diagrammatic representation—are also equal for qutrits.

We can easily generalise this construction to an arbitrary number of qutrits:



We can also define more general phase gadgets where the phases don't have to be related to each other, i.e. we can replace $Z(\alpha, 2\alpha)$ with $Z(\alpha, \beta)$. In this case we would still be calculating the value $x + y + z + w$ modulo 3, but then we apply a different phase depending on the value of this sum: if it is 0 we don't apply any phase; if it is 1 we apply ω^α ; and if it is 2 we apply ω^β .

A particularly relevant choice of phases here is when $\alpha = \beta$. In this case, we apply the phase iff the sum value is not 0. For a trit x it turns out that $x^2 = 0$ if $x = 0$ and $x^2 = 1$ otherwise — this is actually a consequence of Fermat's little theorem and generalises to $x^{p-1} = 1$ modulo p when $x \neq 0$ for p prime. Hence:

$$\text{---} \left(\frac{\alpha}{a} \right) \text{---} :: |x\rangle \mapsto \omega^{\alpha(x^2 \bmod 3)} |x\rangle \quad (9)$$

There is a complication with the phase gadget circuit representation that doesn't arise in the qubit setting, which is that the CNOT gate is self-inverse while the CX qutrit gate is not. In Eq. (6) we needed

to pair a CX with a CX^\dagger to make the construction work. If we instead have a pair of CX gates, we get something a bit more complicated:

$$(10)$$

Remark 4.1. Another way to view qubit phase gadgets is as an exponentiated Pauli $e^{i\alpha Z \otimes Z}$ [18, 19]. This however does *not* generalise to qutrits, as the qutrit Pauli Z is not self-adjoint, and hence cannot be exponentiated to give a unitary. In fact, a qutrit phase gadget cannot be represented as the exponential of a ‘pure tensor’ like $e^{i\alpha A \otimes B}$. This does suggest that there could be another suitable generalisation of a phase gadget that is the exponential of a tensor of *Gell-Mann matrices*, a qutrit basis of self-adjoint matrices.

4.2 Controlled phase gates

The other type of useful diagonal gate for qubits is the *controlled phase gate*. Such a gate applies a $Z(\alpha)$ gate on a qubit controlled on the value of a control. There are multiple ways in which we can generalise these to the qutrit setting. The type of control we will consider first is the $|2\rangle$ -control of Definition 2.8. To see how we can build a $|2\rangle$ -controlled Z phase gate, we will take inspiration from the qubit construction. Recall that there we have:

$$(11)$$

We can ‘port’ the right-hand side to the qutrit setting, by taking each of the phases to be a $Z(\alpha, \beta)$. However, we then run into some problems. It is easy to check that when the top qutrit (the control) is $|0\rangle$ that the diagram indeed acts as the identity on the bottom qutrit (the target). However, it implements a different phase gate on the target depending on whether the control is in $|1\rangle$ or $|2\rangle$:

$$(12)$$

Seeing as we want to construct the $|2\rangle$ -controlled gate that should act as the identity when the control is $|1\rangle$ this is a problem. We solve this issue by ‘doubling up’ the construction, with the second construction being conjugated by X_{12} on the control in order to interchange the role of $|1\rangle$ and $|2\rangle$:

$$(13)$$

By referring to Eq. (12) we see then that in order for Eq. (13) to be equal to the $|2\rangle$ -controlled $Z(\theta, \phi)$ gate it needs to satisfy a set of linear equations. We can solve these to get a (unique up to some Clifford phases) solution:

$$(14)$$

We can hence write any $|2\rangle$ -controlled phase gate using at most four CX gates and four phase gates. For example, if we pick $\theta = 1$ and $\phi = 2$ (so that we are constructing the controlled Z gate) we get:

(15)

Here we write this blue dot with a 2 in it to denote a $|2\rangle$ -control. We see then that our construction in the special case of $|2\rangle$ -controlled Paulis indeed achieves the lowest known T -count of 3 [9]. By conjugating the control wire by either X_{+1} or X_{-1} we can make the gate also be controlled on either $|1\rangle$ or $|0\rangle$.

We can add any number of controls to our construction in Eq. (13) to make it controlled on any tritstring. Without loss of generality, let us say the tritstring in question is $|2\rangle^{\otimes n}$ (by conjugating with X gates, we can make this into a control on any tritstring of length n). The naïve way to construct this controlled gate is to inductively add controls to each gate in the decomposition: controlled constructions for the X or CX gate are described in, for instance, Ref. [56], while controlled Z phase gates can be constructed by recursively applying Eq. (13). However, this method is not efficient as it requires an exponential number of gates as the number of controls increases.

We can do better by not adding controls to all the gates in the decomposition:

(16)

In the case where all the controlled gates fire, this indeed implements any desired Z phase gate on the target qutrit. Otherwise, none of the controlled gates fire, and then the bottom two qutrits becomes identity (use $(H2)$ and $(H4)$ on the top qutrit and (SZ) and (ID) on the bottom qutrit). We hence get the following proposition.

Proposition 4.2. Any tritstring-controlled qutrit Z or X phase gate can be constructed without ancillae and with a polynomial number of Clifford and phase gates.

Proof. The X phase gates can be constructed from the Z phase gates by conjugating by Hadamards, so we only need to describe how to construct tritstring-controlled Z phase gates. Suppose we wish to construct a phase gate with n controls. By our prior work in Ref. [56], each $|2\rangle^{\otimes(n-1)}$ -controlled CX gate can be built ancilla-free using $O(n^{2.585})$ qutrit Clifford+ T gates. It then remains to show how to construct the $|2\rangle^{\otimes(n-1)}$ -controlled Z phase gate in Eq. (16). We do this recursively. To construct the gate with k controls, we need four controlled CX gates with $k - 1$ controls and a $k - 1$ controlled Z phase gate, which then needs four controlled CX gates with $k - 2$ controls, and so on. The total asymptotic gate count is then $4O(n^{2.585}) + 4O((n - 1)^{2.585}) + \dots$ which gives us a gate count of $O(n^{3.585})$. \square

Note that in this construction, the size of the phases involved becomes exponentially smaller in the number of controls. We will next see that there is a notion of control which circumvents this issue.

4.3 Phase multipliers

The $|2\rangle$ -controlled phase gate is just one possible way to extend the idea of a controlled-phase gate from qubits. Another way is to realise that for qubits we can describe the action of a controlled phase gate as $|x, y\rangle \mapsto e^{i\alpha \cdot x \cdot y} |x, y\rangle$. Indeed, if the control qubit is in the state $x = 0$, then this is just the identity, while if $x = 1$, we apply $e^{i\alpha y}$ which corresponds to a $Z(\alpha)$ gate on the $|y\rangle$ qubit.

We see then that while a phase gadget is based on the addition operation of \mathbb{Z}_2 , controlled phase gates are based on the multiplication operation of \mathbb{Z}_2 . This suggests that the controlled phase gate equivalent for qutrits should be $|x, y\rangle \mapsto e^{i\alpha x \cdot y} |x, y\rangle$ where now we take $x \cdot y$ modulo 3. We will show how we can construct this operation using phase gadgets. In order to distinguish this type of gate from the previously considered controlled phase gates, we will refer to a gate where the phase depends on $x \cdot y$ as a *phase multiplier*. Before we show how to build phase multipliers for qutrits, we first need to understand how to build them for qubits. For bits x and y we have the relation

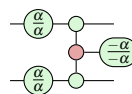
$$x \cdot y = \frac{1}{2}(x + y - (x \oplus y)). \quad (17)$$

Importantly, we are considering the $+$ operation here not modulo 2, but just as an action on real numbers, and we are writing \oplus for addition modulo 2. Using this relation we can write $e^{i\alpha(x \cdot y)} = e^{i\frac{1}{2}\alpha(x+y-(x \oplus y))} = e^{i\frac{1}{2}\alpha x} e^{i\frac{1}{2}\alpha y} e^{-i\frac{1}{2}\alpha(x \oplus y)}$. This is where the circuit decomposition of Eq. (11) comes from. This relation between additive and multiplicative phase gates follows from a Fourier-type duality that exists for semi-Boolean functions, which is explored in detail in Ref. [39].

It turns out that a similar decomposition is possible for qutrits. Note that we can derive Eq. (17) by starting with the equation $(x + y)^2 = x^2 + y^2 + 2x \cdot y$ and then realising that $x^2 = x$ for $x \in \{0, 1\}$ so that this reduces to $x \oplus y = x + y + 2x \cdot y$ for bits. When working with trits we can't remove these squares, but we can still get a useful relation. Bring terms to the other side to get $-2x \cdot y = x^2 + y^2 - (x + y)^2$ and then use the fact that modulo 3 we have $-2 = 1$ to get $x \cdot y = x^2 + y^2 - (x + y)^2$. It is now straightforward to check that this continues to hold when we interpret the outer $+$ and $-$ here not modulo 3, but as operations on the real numbers, so that we get the relation:

$$x \cdot y \bmod 3 = (x^2 \bmod 3) + (y^2 \bmod 3) - ((x + y)^2 \bmod 3) \quad (18)$$

Hence, using Eq. (9) we get the following decomposition:



$$|x, y\rangle \mapsto \omega^{\alpha(x \cdot y \bmod 3)} |x, y\rangle \quad (19)$$

We can easily generalise Eq. (18) to as many variables as desired by iterating it. For three trits:

$$\begin{aligned} (x \cdot y) \cdot z &= x^2 \cdot z + y^2 \cdot z - (x + y)^2 \cdot z \\ &= x^4 + z^2 - (x^2 + z)^2 + y^4 + z^2 - (y^2 + z)^2 - (x + y)^4 - z^2 + ((x + y)^2 + z)^2 \\ &= x^2 + y^2 + z^2 - (x^2 + z)^2 - (y^2 + z)^2 - (x + y)^2 + ((x + y)^2 + z)^2 \end{aligned} \quad (20)$$

Here we used that $x^4 = x^2$ modulo 3.

Note that Eq. (9) shows how to apply a phase proportional to the input trit squared modulo 3. However, in order to use this trick to apply a phase proportional to a higher order term such as $(y + x^2)^2$, we need a way to compute $y + x^2$ and store it “on the wire”. In other words, we need to construct a circuit for the unitary defined by $|x, y\rangle \mapsto |x, y + x^2\rangle$. Because this simply adds 1 (modulo 3) to y iff $x \neq 0$, we construct

it by adding 1 to the second qubit, and then applying a $|0\rangle$ -controlled X_{-1} gate. To build this gate, we use the $|2\rangle$ -controlled Z gate that we built from two phase gadgets in Eq. (15):

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \begin{array}{c} \textcircled{0} \\ | \\ \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \begin{array}{c} X_{-1} \textcircled{2} X_{+1} \\ | \\ H^\dagger Z^\dagger Z H \end{array} \quad \because \quad |x, y\rangle \mapsto |x, y + x^2\rangle \quad (21)$$

The above trick was also described in Ref. [9]. We further use this to build the type of phase gate below.

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \begin{array}{c} \textcircled{0} \\ | \\ \text{---} \text{---} \end{array} \begin{array}{c} \textcircled{\alpha} \\ | \\ \text{---} \text{---} \end{array} \begin{array}{c} \textcircled{0} \\ | \\ \text{---} \text{---} \end{array} \quad \because \quad |x, y\rangle \mapsto \omega^{\alpha(y+x^2)^2} |x, y\rangle. \quad (22)$$

We now have all the ingredients necessary to build the phase multiplier corresponding to the formula (20). What is interesting about this is that we do not have to use smaller factors of α . This is in contrast to the qubit counterpart of the formula (20) where we get a factor of $\frac{1}{4}$, due to the factor of $\frac{1}{2}$ in Eq. (17). In fact, for qubits, the generalisation to n variables will have a prefactor of $(1/2)^{n-1}$ so that for instance the three-qubit-controlled Z and controlled T gates cannot be constructed without ancillae in Clifford+ T [25], as we need $\pi/8$ phase gates. Instead, no matter the number of qutrits, we do not get such a prefactor and can iteratively construct it as in the formula (20), as we did for qutrit Clifford+ T in Ref. [56]. The circuit (22), alongside the square phase of Eq. (9) suffices to generalise Eq. (19) to any number of qutrits.

Proposition 4.3. We can construct, without ancillae and using $O(2^n)$ Clifford+ T , $Z(\alpha, \alpha)$, and $Z(-\alpha, -\alpha)$ gates, the n -qutrit phase multiplier gate defined by $|x_1, \dots, x_n\rangle \mapsto \omega^{\alpha((x_1 \dots x_n) \bmod 3)} |x_1, \dots, x_n\rangle$.

See Appendix B for the details.

Remark 4.4. In Ref. [21] the diagonal gates at all levels of the Clifford hierarchy are analysed for any qudit of prime dimension. They show for instance that the gate implementing $|x_1 \dots x_n\rangle \mapsto \omega^{x_1 \dots x_n} |x_1 \dots x_n\rangle$ (which is the n -controlled $2\pi/3$ phase multiplier gate) is in the n th level of the Clifford hierarchy. This might be surprising as our construction shows how to build this gate, for any n , only using gates from the third level of the Clifford hierarchy (namely Clifford gates and the T gate). However, note that while the *diagonal* gates on a level of the hierarchy form a group, the full set of (not necessarily diagonal) gates is not closed under composition, and hence we can build higher-level unitaries using lower-level ones.

5 Applications

We've now seen that we can use phase gadgets to build a number of useful diagonal unitaries. In this section we will see how we can build more general diagonal qutrit unitaries, and specifically those that *emulate* qubit operations. Qudit emulation of qubit operations can result in efficiency gains, by using higher level states rather than ancillae. While there has been significant work on emulating qubits using qutrits and qudits, much of this has been limited to realising gates within classical reversible computing such as multiple-controlled Toffolis. In contrast, fewer works have addressed qutrit gate sets containing arbitrary phases. Examples include a $|2\rangle$ -controlled $Z(0, \phi)$ decomposition in terms of qutrit-controlled qubit $\phi/3$ rotations [24] or quantum multiplexers and uniformly-controlled Givens rotations from the cosine-sine decomposition [37].

Throughout this section, we will write $\stackrel{e}{=}$ to denote that a qubit unitary is emulated by a qutrit unitary.

We will first see how to emulate arbitrary qubit diagonal unitaries. Note that when we restrict to the $\{|0\rangle, |1\rangle\}$ subspace, that a qutrit phase multiplier $|x_1, \dots, x_n\rangle \mapsto e^{i\alpha((x_1 \dots x_n) \bmod 3)} |x_1, \dots, x_n\rangle$ only applies a

phase α if and only if all n qubits are in the $|1\rangle$ state. Hence, for instance, the two-qubit $\text{CZ}(\alpha)$ gate is directly emulated by its two-qutrit counterpart of Eq. (19) with action $|x, y\rangle \mapsto e^{i\alpha x \cdot y} |x, y\rangle$. Consequently, by Proposition 4.3 we see that using qutrit Clifford+ T gates along with $Z(\alpha, \alpha)$ and $Z(-\alpha, -\alpha)$ we can emulate the multiple-controlled $Z(\alpha)$ qubit gate without ancillae.

Now, by conjugating a multiple-controlled $\text{CZ}(\alpha)$ gate by the appropriate X_{01} gates, we can decide on which input the $e^{i\alpha}$ phase is applied. Using multiple of these gates we can then arbitrarily decide for each input which phase should be applied to it. This then allows us to emulate an arbitrary diagonal qubit gate.

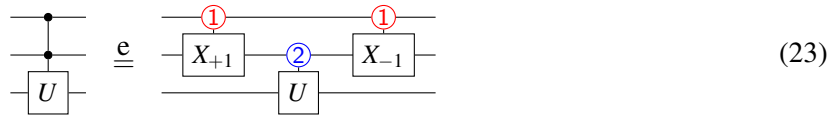
Proposition 5.1. We can emulate the $\text{diag}(\omega^{\alpha_1}, \dots, \omega^{\alpha_{2^n}})$ qubit unitary using qutrit Clifford+ T , $Z(\alpha_j, \alpha_j)$ and $Z(-\alpha_j, -\alpha_j)$ gates and without using ancillae.

When using a standard qubit unitary synthesis algorithm, the desired phases $e^{i\alpha_j}$ would be implemented using many-controlled phase gates that require exponentially small angles $e^{i\alpha_j/2^n}$, which is problematic when the use-case is in fault-tolerant computing where non-Clifford phase gates must be constructed using magic state distillation and injection. Our construction could hence lead to some benefits in synthesising diagonal qubit unitaries using less non-Clifford resources.

It turns out that for the specific case of a qubit CCZ gate, that we can emulate it using qutrits in an even more efficient way. While we could use the emulation construction above, it turns out to be better to consider an altered construction.

Lemma 5.2. Given a qutrit $|2\rangle$ -controlled U gate for an emulated qubit unitary U , we can construct a qutrit emulation of the qubit CCU gate with the same non-Clifford cost as the $|2\rangle$ -controlled U gate.

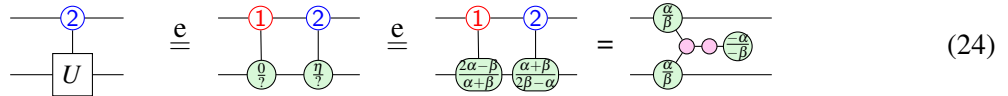
Proof. One can readily verify, by initializing the top two qubits to $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, that the below qutrit decomposition of Ref. [27] emulates the qubit CCU gate.



We can then replace the two non-Clifford $|1\rangle$ -controlled X_{+1} and $|1\rangle$ -controlled X_{-1} by CX and CX^\dagger , preserving correctness of the emulation as the action on the $\{|0\rangle, |1\rangle\}$ subspace is unchanged [10]. \square

Using this lemma we see that to get an efficient emulation of the qubit CCZ, it remains to find an efficient qutrit emulation of the $|2\rangle$ -controlled qubit Z gate.

Lemma 5.3. Let $U = \text{diag}(1, \omega^\eta)$ be an arbitrary qubit Z phase gate. Then we can build the $|2\rangle$ -controlled emulated U using the controlled phase gate of Eq. (12):



Here α and β satisfy, for some $k \in \mathbb{Z}$, $2\alpha - \beta = 3k$ and $\alpha + \beta = \eta$ and the questionmarks ? denote that these phases are irrelevant for the emulation.

If we choose $\alpha = 3/2$ and $\beta = 0$ in this construction we are emulating the $|2\rangle$ -controlled qubit Z gate, because the phase of $\omega^{3/2} = -1$ applies iff the control qutrit is $|2\rangle$ and the target qubit is $|1\rangle$. Note that a $Z(3/2, 0)$ phase is equal to $X_{12} R X_{12}$, referring to the R gate from Definition 2.7. Hence:

Corollary 5.4. The $|2\rangle$ -controlled qubit Z gate can be emulated with R -count 3.

Combine this corollary with Lemma 5.2 we arrive at our result.

Proposition 5.5. The qubit CCZ gate can be emulated ancilla-free in qutrit Clifford+ R with R -count 3.

As shown in Ref. [35], any implementation of a CCZ gate requires at least four qubit T gates. Additionally any unitary implementation requires at least seven [29]. Here we see that surprisingly, by embedding the CCZ into qutrit space, we can construct it using just three non-Clifford single-qutrit gates and that moreover this is unitary and ancilla-free. This is also a new minimum amongst qudit emulations: for instance, in Ref. [33], they needed four qudit (for prime $d > 3$) T gates to emulate qubit CCZ.

6 Conclusion

We introduced phase gadgets in the qutrit ZX-calculus. To do this, we adapted the original qutrit ZX-calculus to be flexsymmetric so that the phase gadgets' behaviour would not depend on the directionality of their edges. Using phase gadgets we showed how to build two types of qutrit controlled phase gates: tritstring-controlled phase gates and phase multipliers. This allowed us to emulate the qubit CCZ gate using just three single-qudit non-Clifford gates.

While some of our constructions will naturally generalise to arbitrary qudit dimension, some things are qutrit specific. It seems to be a coincidence that for qutrits, in contrast with other-dimensional qudits, you can derive a relation between modular multiplication and addition (18) from the same binomial as for qubits (17), which comes from having a natural way to express $x^2 \bmod 3$ thanks to Fermat's little theorem. As a result, qubit and qutrit phase multipliers admit constructions which are structurally similar, despite the fact that for qubits it applies a phase of α on only one possible input — where all n qubits are $|1\rangle$ — while for qutrits it applies a phase, which can be α or 2α , for 2^n of the 3^n possible input basis states. Moreover, it seems quite special that Eq. (18) does not have any factors making the size of the phases internal to the decomposition decrease (in contrast to the qubit case).

We believe we could use these results as a stepping stone towards defining a qutrit ZH-calculus [4]. In the qubit ZH-calculus, the H-boxes represent matrices with coefficients $a^{i_1 \dots i_m j_1 \dots j_n}$ for a complex number a and $i_1, \dots, i_m, j_1, \dots, j_n \in \{0, 1\}$. Therefore, the obvious generalisation to qutrits (at least for a a complex phase) corresponds to our qutrit phase multipliers. Phase gadgets and phase multipliers could then be related in the same way as they are for qubit ZX and ZH [39].

An open problem is to find a suitable qutrit equivalent of exponentiated Paulis. The canonical self-adjoint generalisation of qubit Paulis to qutrits, the Gell-Mann matrices, can be exponentiated to unitaries, but it is not clear how they are related to the qutrit Paulis exactly. A starting point to find the proper relation here is to express exponentiations using a Hermitian operator basis constructed from the qutrit Paulis [3].

Finally, let us mention that based on work on an earlier draft of this paper, a proposed scheme for physically implementing a qutrit phase gadget in superconducting qutrit hardware was made [14].

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A Qutrit ZX-calculus

A.1 Necessity of rules

We can show that most of the rules in Figure 1 are *necessary*, meaning that they cannot be derived from the other rules. We do this by adapting the reasoning of Ref. [49]. Namely, the following rules are definitely necessary:

- (SZ): this is the only rule which can decompose a generator with four or more legs into generators with fewer legs.
- (P): this is the only rule which resolves diagrams containing generators to the identity.
- (B1): this is the only rule that can transform a connected diagram into a disconnected one.
- (EU): this is necessary per the argument of Ref. [51, Proposition 3.2].
- At least one of (H) and (H') is necessary as these are the only ones that can convert a diagram containing a X generator with a non-integer phase into one containing a Z generator with a non-integer phase.

We do not know whether the other rules are necessary, although we do suspect this is the case.

A.2 Proofs of the derived rules

Lemma A.1. The (ID) rule can be derived from the (SZ) and (SP) rules.

Proof.

$$\text{---} \circ \text{---} \stackrel{(P)}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{(SZ)}{=} \text{---} \circ \text{---} \stackrel{(SP)}{=} \text{---} \quad \square$$

Lemma A.2. The (H2) rule can be derived from the (H') and (ID) rules.

Proof.

$$\text{---} \text{---} \text{---} \stackrel{(ID)}{=} \text{---} \circ \text{---} \stackrel{(H')}{=} \text{---} \circ \text{---} \quad \square$$

Lemma A.3. The (H4) rule can be derived from the (H), (ID), and (H2) rules.

Proof.

$$\text{---} \text{---} \text{---} \text{---} \stackrel{(H2)}{=} \text{---} \circ \text{---} \stackrel{(H)}{=} \text{---} \circ \text{---} \stackrel{(ID)}{=} \text{---} \quad \square$$

Lemma A.4. The (SX) rule can be derived from the (SZ), (H'), (H2), and (H4) rules.

Proof.

$$\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \stackrel{(H')}{=} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \stackrel{(H2)}{=} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \stackrel{(H4)}{=} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \stackrel{(SZ)}{=} \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \stackrel{(H')}{=} \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \square$$

Lemma A.5. The (P1') rules can be derived from the (P1), (SX), (H), (H'), (H2), and (H4) rules.

Proof. Let's first derive the rule for $x = 0$:

$$\begin{array}{c} \text{(H2)} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{(H4)} \\ \text{---} \text{///} \end{array} = \begin{array}{c} \text{(H')} \\ \text{---} \text{///} \end{array} = \begin{array}{c} \text{(H)} \\ \text{---} \text{///} \end{array} = \begin{array}{c} \text{(H2)} \\ \text{---} \circ \text{---} \end{array} \quad (25)$$

From that, we can derive the below rule:

$$\begin{array}{c} \text{(SX)} \\ \text{---} \left(\frac{2}{i} \right) \end{array} = \begin{array}{c} \text{(P1)} \\ \text{---} \left(\frac{1}{2} \right) \circ \left(\frac{1}{2} \right) \end{array} = \begin{array}{c} \text{(25)} \\ \text{---} \left(\frac{1}{2} \right) \text{///} \end{array} = \begin{array}{c} \text{(P1)} \\ \text{---} \left(\frac{1}{2} \right) \text{///} \end{array} = \begin{array}{c} \text{(SX)} \\ \text{---} \left(\frac{2}{i} \right) \end{array} \quad (26)$$

The two above rules, along with the (P1) rule, are captured by the following rule where $x \in \{0, 1, 2\}$:

$$\begin{array}{c} \text{---} \left(\frac{x}{2x} \right) \end{array} = \begin{array}{c} \left(\frac{x}{2x} \right) \end{array} \quad (27)$$

We now colour-change the above rule to finish deriving all the (P1') rules:

$$\begin{array}{c} \text{(H)} \\ \text{---} \left(\frac{x}{2x} \right) \end{array} = \begin{array}{c} \text{(H')} \\ \text{---} \left(\frac{x}{2x} \right) \text{///} \end{array} = \begin{array}{c} \text{(H2)} \\ \text{---} \left(\frac{x}{2x} \right) \text{///} \end{array} = \begin{array}{c} \text{(27)} \\ \text{---} \left(\frac{x}{2x} \right) \end{array} = \begin{array}{c} \text{(H')}, \\ \text{(H2)} \\ \text{---} \left(\frac{2x}{x} \right) \text{///} \end{array} = \begin{array}{c} \text{(H')}, \\ \text{(H4)} \\ \text{---} \left(\frac{2x}{x} \right) \end{array} \quad (28)$$

□

Lemma A.6. The (P2') rules can be derived from the (P2), (H), (H'), (SZ), (SX), (H2), and (H4) rules.

Proof. We prove them one by one:

$$\begin{array}{c} \text{(SX)} \\ \left(\frac{2}{i} \right) \left(\frac{\alpha}{\beta} \right) \end{array} = \begin{array}{c} \text{(H2)}, \\ \text{(P1)} \\ \left(\frac{1}{2} \right) \left(\frac{\alpha}{\beta} \right) \end{array} = e^{i\alpha} \begin{array}{c} \text{(H)}, \\ \text{(H')} \\ \left(\frac{1}{2} \right) \text{///} \end{array} = e^{i\alpha} \begin{array}{c} \text{(H4)}, \\ \text{(P1)} \\ \left(\frac{1}{2} \right) \text{///} \end{array} = e^{i(\alpha+\beta-\alpha)} \begin{array}{c} \text{(H2)}, \\ \text{(SX)} \\ \left(\frac{-\beta-\alpha}{-\alpha-\beta-\alpha} \right) \left(\frac{1}{2} \right) \end{array} = e^{i\beta} \begin{array}{c} \left(\frac{\alpha-\beta}{-\beta} \right) \left(\frac{2}{i} \right) \end{array} \quad (29)$$

$$\begin{array}{c} \text{(H)} \\ \left(\frac{1}{2} \right) \left(\frac{\alpha}{\beta} \right) \end{array} = \begin{array}{c} \text{(H4)}, \\ \text{(H)} \\ \left(\frac{1}{2} \right) \left(\frac{\alpha}{\beta} \right) \text{///} \end{array} = \begin{array}{c} \text{(K2)} \\ e^{i\alpha} \text{---} \left(\frac{-\alpha}{\beta-\alpha} \right) \left(\frac{1}{2} \right) \text{///} \end{array} = \begin{array}{c} \text{(H')}, \\ \text{(H4)} \\ e^{i\alpha} \text{---} \left(\frac{-\alpha}{\beta-\alpha} \right) \left(\frac{2}{i} \right) \end{array} = e^{i\alpha} \begin{array}{c} \left(\frac{-\beta-\alpha}{-\beta-\alpha} \right) \left(\frac{2}{i} \right) \end{array} \quad (30)$$

$$\begin{array}{c} \text{(SZ)} \\ \left(\frac{2}{i} \right) \left(\frac{\alpha}{\beta} \right) \end{array} = \begin{array}{c} \text{(30)} \\ \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{\alpha}{\beta} \right) \end{array} = e^{i\alpha} \begin{array}{c} \text{(30)} \\ \left(\frac{1}{2} \right) \left(\frac{-\beta-\alpha}{-\beta-\alpha} \right) \left(\frac{2}{i} \right) \end{array} = e^{i(\alpha+\beta-\alpha)} \begin{array}{c} \left(\frac{-\alpha-\beta-\alpha}{-\beta-\alpha} \right) \left(\frac{2}{i} \right) \end{array} = e^{i\beta} \begin{array}{c} \left(\frac{-\beta}{\alpha-\beta} \right) \left(\frac{1}{2} \right) \end{array} \quad (31)$$

□

Lemma A.7. The (EU') rules can be derived from the (EU), (H), (H'), (SZ), (SX), (H2), and (H4) rules.

Proof. We show the first equation directly:

$$\begin{array}{c} \text{(EU)} \\ \text{---} \text{///} \end{array} = i \begin{array}{c} \text{(H)} \\ \text{---} \left(\frac{2}{3} \right) \left(\frac{2}{3} \right) \left(\frac{2}{3} \right) \end{array} = i \begin{array}{c} \text{(H)} \\ \text{---} \left(\frac{2}{3} \right) \text{///} \end{array} = i \begin{array}{c} \text{(H)} \\ \text{---} \left(\frac{2}{3} \right) \left(\frac{2}{3} \right) \left(\frac{2}{3} \right) \end{array} \quad (32)$$

phase $e^{i\alpha(t_j^2+w)^2}$. This construction involves temporarily storing $t_j^2 + w$ on the wire of w , so we can use this term if we want to construct the five-qutrit phase multiplier as well.

We then see that the cost of the n -qutrit phase multiplier in terms of (non-Clifford) gates is the cost of the $n - 1$ qutrit phase multiplier plus the cost of $2^{n-1} - 1$ applications of the Eq. (22) gadget. In particular, each phase term requires precisely one of either a $Z(\alpha, \alpha)$ or a $Z(-\alpha, -\alpha)$ gate, so that we need $2^n - 1$ of them. The circuit of Eq. (22) requires 6 T gates to construct, and hence the T -count of the n -qutrit phase multiplier is $6(2^{n-1} - 1) = 3 \cdot 2^n - 6$ (for $n > 2$).