

Logical Characterization of Contextual Hidden-Variable Theories Based on Quantum Set Theory

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While non-contextual hidden-variable theories are proved to be impossible, contextual ones are possible. In a contextual hidden-variable theory, an observable is called a beable if the hidden-variable assigns its value in a given measurement context specified by a state and a preferred observable. Halvorson and Clifton characterized the algebraic structure of beables as a von Neumann subalgebra, called a beable subalgebra, of the full observable algebra such that the probability distribution of every observable affiliated therewith admits the ignorance interpretation. On the other hand, we have shown that for every von Neumann algebra there is a unique set theoretical universe such that the internal “real numbers” bijectively correspond to the observables affiliated with the given von Neumann algebra. Here, we show that a set theoretical universe is associated with a beable subalgebra if and only if it is ZFC-satisfiable, namely, every theorem of ZFC set theory holds with probability equal to unity. Moreover, we show that there is a unique maximal ZFC-satisfiable subuniverse “implicitly definable”, in the sense of Malament and others, by the given measurement context. The set theoretical language for the ZFC-satisfiable universe, characterized by the present work, rigorously reconstructs Bohr’s notion of the “classical language” to describe the beables in a given measurement context.

1 Introduction

In 1935, Einstein, Podolsky, and Rosen [14] argued that the quantum mechanical description of physical reality is incomplete. Bohr [6] immediately responded to rebut their conclusion. As to which claim is correct, the majority view has become in favor of Bohr, based on no-go theorems against non-contextual hidden-variable theories by von Neumann [25], Gleason [16], Kochen-Specker [22]. From the above debate it has been concluded that non-contextual hidden-variable theories for quantum mechanics are impossible. Nevertheless, contextual hidden-variable theories are possible as Bohr’s complementarity interpretation [5] and the Bohmian mechanics [4].

In accordance with the above view, the modal interpretation of quantum mechanics has been studied extensively, used for no-collapse interpretation to solve the measurement problem [9], and successfully articulated Bohr’s otherwise obscure complementarity interpretation of quantum mechanics [9,18,33]. In modal interpretation, an observable possessing a well defined value is called a “beable”, and it is attempted to define the class of beables as broad as possible depending on the given measurement context. The observable to be measured is naturally considered to possess its value to be revealed by the measurement even in a superposition of its eigenstates. This avoids the measurement problem arising from state collapsing with the Dirac-von Neumann interpretation [12,25] that an observable has a value only in its eigenstates or mixtures of them [9].

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Halvorson and Clifton [17] algebraically characterized a contextual hidden-variable theory for a state ψ in the Hilbert space \mathcal{H} of a quantum system \mathbf{S} as a von Neumann subalgebra \mathcal{B} , called a beable subalgebra, of the full observable algebra $\mathcal{L}(\mathcal{H})$ such that the probability distribution of every observable therein admits the ignorance interpretation. Their celebrated uniqueness theorem determines the unique maximal beable subalgebra $\mathcal{B}(\psi, A)$ “implicitly definable”, in the sense of Malament [23] and others, by the given measurement context (ψ, A) . However, the algebraic structure of beables does not directly treat the logical structure of observational propositions nor the structure of the language speaking of beables, so that we do not have a formal framework to treat, for instance, Bohr’s original notion of the “classical language” to describe beables in a given measurement context, or Hardy’s logical formulation of non-locality.¹

Here, we introduce a new approach based on quantum set theory to provide a logical framework for modal interpretations. Quantum set theory was introduced by Takeuti [37] for constructing mathematics based on quantum logic and developed by the present author [26–32]; the relationship with topos quantum theory was studied by Eva [15] and Döring *et al.* [13]. In the preceding study, we have shown that for any von Neumann subalgebra \mathcal{M} of the full observable algebra $\mathcal{L}(\mathcal{H})$ on a Hilbert space \mathcal{H} , we can construct the unique mathematical, or more specifically, set theoretical, universe $V[\mathcal{M}]$ based on the logic represented by the projection lattice $\mathcal{P}(\mathcal{M})$ in \mathcal{M} such that the internal “real numbers” in the universe $V[\mathcal{M}]$ coincides with the self-adjoint operators (or observables) affiliated with the von Neumann algebra \mathcal{M} .

In this work, we shall logically characterize a contextual hidden-variable theory by showing that a von Neumann algebra \mathcal{B} is a beable subalgebra of $\mathcal{L}(\mathcal{H})$ for a state $\psi \in \mathcal{H}$ if and only if the set theoretical universe $V[\mathcal{B}]$ based on the logic $\mathcal{P}(\mathcal{B})$ is ZFC-satisfiable in ψ , in the sense that every theorem of ZFC set theory in the language $\mathbf{L}(\in, V[\mathcal{B}])$ of set theory augmented by the names of elements of $V[\mathcal{B}]$ holds in $V[\mathcal{B}]$ with probability equal to unity in the state ψ . In this case, the set of beables represented by self-adjoint operators affiliated with \mathcal{B} coincides with the set of the internal “real numbers” in the universe $V[\mathcal{B}]$. Moreover, we uniquely determine the maximal ZFC-satisfiable subuniverse $V[\mathcal{B}(\psi, A)]$ among those implicitly definable by the given measurement context (ψ, A) . Thus, we can identify Bohr’s notion of “classical language” describing the beables in the given measurement context (ψ, A) with the language $\mathbf{L}(\in, V[\mathcal{B}(\psi, A)])$.²

2 Algebraic approach to beables

In this paper, we consider a quantum system \mathbf{S} described by a separable Hilbert space \mathcal{H} , called the *state space* of \mathbf{S} , with inner product (ξ, η) for all $\xi, \eta \in \mathcal{H}$, linear in η and conjugate linear in ξ . Observables of \mathbf{S} are bijectively represented by self-adjoint operators (densely defined) on \mathcal{H} and every unit vector in \mathcal{H} represents a (pure) state of \mathbf{S} . If an observable (represented by a self-adjoint operator) X is measured in a state (represented by a unit vector) $\phi \in \mathcal{H}$, the outcome \mathbf{x} of the measurement satisfies the Born statistical formula

$$\Pr\{\mathbf{x} \leq x \mid \phi\} = (\phi, E^X(x)\phi) \quad (1)$$

for every real number $x \in \mathbb{R}$, where $E^X(x)$ is the resolution of the identity belonging to the self-adjoint operator X [25, p. 119]. We denote by μ_ϕ^X the Borel probability measure on \mathbb{R} uniquely determined by

¹We focus on the former topic in this paper and we will discuss the latter elsewhere.

²Bohr called the notion of beables in the given measurement context in several different ways, e.g., as the observables definable for the given measurement arrangement, or the elements of physical reality determined by the measurement of a preferred observable in the given state.

the relation $\mu_\phi^X((-\infty, x]) = (\phi, E^X(x)\phi)$, and call it the (*Born*) *probability distribution* of the observable X in the state ϕ . From the above, for any (real-valued) Borel function f the observable $f(X)$ defined by $f(X) = \int_{\mathbb{R}} f(x)dE^X(x)$ has the expectation value

$$\text{Ex}\{f(X)\|\phi\} = (\phi, f(X)\phi) \quad (2)$$

if $\phi \in \text{dom}(f(X))$.

Now we consider a situation, called the *measurement context* (ψ, A) , in which an observable A is to be measured in a state ψ , and we take the ignorance interpretation for the Born probability distribution μ_ψ^A , as a typical reading of Bohr's complementarity interpretation [9,17,18,33], that just before the measurement of the observable A in the state ψ , the observable A possesses its value with the probability distribution μ_ψ^A , and that the measurement faithfully reveals the value possessed by A . We would call an observable that is considered to possess its value in the measurement context (ψ, A) as a *beable* [2, p. 41] in that measurement context. Obviously, the observable A itself should be a beable together with its functions $f(A)$ for all Borel functions f .

The objective of modal interpretations is to determine the set of beables in the context (ψ, A) as broad as possible. We would call it the *maximal beable set*. From the impossibility theorem of non-contextual hidden-variable theories by von Neumann [25] and others,³ the maximal beable set cannot be the whole set of observables.

It is natural to assume that the maximal beable set is closed under appropriate algebraic operations (i.e., addition, Jordan product, and Lie product) and closed under appropriate convergences, so that we assume that the maximal beable set is the set of observables (or self-adjoint operators) affiliated with a von Neumann algebra \mathcal{B} on \mathcal{H} .⁴

The first requirement for such a von Neumann algebra \mathcal{B} concerns the state ψ and requires that the Born probability distribution of every beable admit the ignorance interpretation. This requirement is mathematically formulated as follows. We call any normalized positive linear functional on \mathcal{B} a *state* on \mathcal{B} . A state ω on \mathcal{B} is said to be *dispersion-free* iff $\omega(X^*X) = |\omega(X)|^2$ for any $X \in \mathcal{B}$. We say that a von Neumann algebra \mathcal{B} is a *beable subalgebra for a state vector* $\psi \in \mathcal{H}$ iff there is a probability measure μ on the space $\mathcal{D}(\mathcal{B})$ of dispersion-free states of \mathcal{B} satisfying

$$(\psi, X\psi) = \int_{\mathcal{D}(\mathcal{B})} \omega(X)d\mu(\omega) \quad (3)$$

for every $X \in \mathcal{B}$. The second requirement is, of course, that the observable A be affiliated with \mathcal{B} as a "privileged" observable, from which it follows that $f(A)$ be affiliated with \mathcal{B} for all Borel functions f by the Borel function calculus in von Neumann algebras. Thus, we call a von Neumann algebra \mathcal{B} on \mathcal{H} a *beable subalgebra for the measurement context* (ψ, A) iff it satisfies following conditions:

- (i) (Beable) \mathcal{B} is a beable subalgebra for ψ .
- (ii) (A-Priv) A is affiliated with \mathcal{B} .

According to the above formulation, we are tempted to identify the maximal beable set with the set of observables affiliated with a maximal von Neumann algebra among those satisfying the requirements (A-Priv) and (Beable) above. However, such a choice is not unique.

³The first general proof of the impossibility theorem for non-contextual hidden-variable theories was given by von Neumann [25]; see [1,10,11,24] for the recent debate on the status of von Neumann's impossibility proof. Later, Kochen and Specker [22] proved the theorem for the Hilbert space with the dimension greater than 2 under the sole requirement that hidden-variables satisfy functional relations for observables; a similar result can be derived as a corollary of Gleason's theorem [16].

⁴An observable X is affiliated with \mathcal{B} if and only if $E^X(x) \in \mathcal{B}$ for all $x \in \mathbb{R}$.

To see this, consider the measurement context (ψ, A) for the composite system \mathbf{S} of two spin 1/2 particles with the state space $\mathcal{H} = \mathbf{C}^2 \otimes \mathbf{C}^2$, consisting of the singlet state $\psi = 2^{-1/2}(|+_z\rangle|-_z\rangle - |-_z\rangle|+_z\rangle)$ and the z -component $A = \sigma_z \otimes I$ of Pauli spin operators of the first particle. In this case, we have many beable subalgebras $\mathcal{B}_\theta = W^*(\sigma_z) \otimes W^*(\sigma_\theta)$, where $\sigma_\theta = \cos \theta \sigma_z + \sin \theta \sigma_x$, for $0 \leq \theta < \pi$.⁵ Yet, there is no common maximal subalgebra \mathcal{B}_{\max} , since if \mathcal{B}_{\max} were to include \mathcal{B}_θ and $\mathcal{B}_{\theta'}$ with $\theta \neq \theta'$, there would be no dispersion-free state on \mathcal{B}_{\max} .

To consider which beable subalgebra we should choose, recall the debate between EPR [14] and Bohr [6] around the “reality criterion” proposed by EPR.

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity. [14, p.777]

EPR argued, in accordance with this criterion, that in the EPR state the position and momentum of the second particle have simultaneous reality, because the measurement on the first particle measures the second particle without disturbing it [36, p. 334]. Yet, the position and momentum cannot have simultaneous reality in any states by the uncertainty principle, so that EPR concluded that quantum-mechanical description of physical reality is incomplete. Bohr immediately responded to EPR. Bohr claimed that although the measurement on the first particle does not mechanically disturb the second particle, the measurement on the first particle influences the condition that defines elements of reality for the second particle, and he rejected EPR’s conclusion.

[T]here is in a case like that just considered no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system*. . . . [W]e see that the argumentation of the mentioned authors [EPR] does not justify their conclusion that quantum-mechanical description is essentially incomplete.” [6, p. 700]

Following the reality criterion posed by EPR [14], but “contextualized” to the particular measurement context (ψ, A) as suggested by Bohr [6] above, we should choose $B_0 = I \otimes \sigma_z$ to be a beable in this measurement context but not $B_\theta = I \otimes \sigma_\theta$ with $\theta \neq 0$, because only the value of B_0 can be inferred from the value of $A = \sigma_z \otimes I$ in the measurement context (ψ, A) without disturbing the second particle. If the observer were to measure the observable $A' = \sigma_x \otimes I$ instead of $A = \sigma_z \otimes I$, then from the value of A' the observer could infer the value of $B_{\pi/2} = I \otimes \sigma_x$ without disturbing the second particle. EPR might have concluded that both B_0 and $B_{\pi/2}$ are beables, or elements of reality. However, Bohr [6] pointed out that the status of being beable depends on the inference from the value of A to the value of B_0 or the inference from the value of A' to the value of $B_{\pi/2}$, but each inference is justified only in the respective context, in which classical logic and classical mathematics can be used in the ordinary sense, and that there is no context-free classical language that supports the above two types of inferences simultaneously. Thus, what are beables of the second particle depends on what is measured on the first particle as Bohr [6] suggested.

Halvorson and Clifton [18, pp.14–15] proposed a mathematical approach to single out appropriate beable subalgebras consistent with the above “contextualized” reality criterion, as follows. We say that a von Neumann algebra \mathcal{B} is *implicitly definable* by a measurement context (ψ, A) iff $U^* \mathcal{B} U = \mathcal{B}$ for any unitary $U \in \mathcal{L}(\mathcal{H})$ such that $U^* A U = A$ and $U \psi = \psi$. In other words, \mathcal{B} is implicitly definable by

⁵For an observable X , we denote by $W^*(X)$ the von Neumann algebra generated by X if X is bounded, or the von Neumann algebra generated by $E^X(x)$ for all $x \in \mathbb{R}$.

(ψ, A) iff the membership relation $X \in \mathcal{B}$ for any $X \in \mathcal{L}(\mathcal{H})$ is not affected (i.e., $X \in \mathcal{B}$ if and only if $\alpha(X) \in \mathcal{B}$) by any automorphism α of $\mathcal{L}(\mathcal{H})$ that leaves ψ and A invariant,⁶ which is widely used in foundational studies, for example, by Malament [23, p.297]. A beable subalgebra \mathcal{B} for (ψ, A) is called *definable* iff it further satisfies

(iii) (Def) \mathcal{B} is implicitly definable by (ψ, A) .

We call a von Neumann algebra \mathcal{B} a *maximally definable beable subalgebra for the measurement context* (ψ, A) iff \mathcal{B} satisfies (A-Priv), (Beable), and (Def) and \mathcal{B} is maximal (in the set inclusion) among all von Neumann subalgebras of $\mathcal{L}(\mathcal{H})$ satisfying those three requirements. Then, the celebrated Halvorson-Clifton uniqueness theorem [17] is stated as follows.

Theorem 2.1 (Halvorson-Clifton [17]). *For any state ψ and an observable A , there exists the unique maximally definable beable subalgebra $\mathcal{B}(\psi, A)$ for the measurement context (ψ, A) , and it is of the form*

$$\mathcal{B}(\psi, A) = W^*(A)P \oplus \mathcal{L}(P^\perp \mathcal{H}), \quad (4)$$

where P is the projection from \mathcal{H} onto the cyclic subspace $\mathcal{C}(\psi, A)$ of \mathcal{H} generated by ψ and A , i.e., $\mathcal{C}(\psi, A) = \{f(A)\psi \mid f \text{ is a bounded Borel function}\}^{\perp\perp}$.

3 Quantum logic

Bohr's view has, unfortunately, prevailed with several improper restatements, and there have been only a few serious attempts to reconstruct his reply in a rigorous analysis. In his early contribution, Howard [19,20] attempted to clarify what is the element of physical reality for Bohr. He focused on Bohr's notion of "classical description". In fact, Bohr emphasizes in several places that one should describe experimental evidence classically.⁷

Howard's view was further sharpened by Halvorson and Clifton [18] in the framework of modal interpretations as presented in the preceding section; see also Ref. [33] for mathematical refinements. While Howard [19,20] formulated Bohr's classicality requirement by the notion of "appropriate" mixture, Halvorson and Clifton reformulate it as the requirement that the Born probability distribution admit the ignorance interpretation. However, Bohr never explained his notion of "classical description" by the ignorance interpretation of the Born probability distribution. In this paper, we attempt to go a step further.

In view of Bohr's writings, it seems that "classical" means classical physics, but he also stated:

[I]t would seem that the recourse to three-valued logic, sometimes proposed as means for dealing with the paradoxical features of quantum theory, is not suited to give a clearer account of the situation, since all well-defined experimental evidence, even if it cannot be analysed in terms of classical physics, must be expressed in ordinary language making use of common logic. [7, p. 317]

⁶Any automorphism α of $\mathcal{L}(\mathcal{H})$ is of the form $\alpha(X) = U^*XU$ for some unitary $U \in \mathcal{L}(\mathcal{H})$ [34, p. 119].

⁷"[I]t is decisive to recognize that, *however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms.* The argument is simply that by the word "experiment" we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangement and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics." [8, p.209]

According to the above, we can see that what Bohr refers to in the word “classical” is a broader concept than classical physics and it is well understood as the classical language that obeys classical logic rather than quantum logic. Therefore, if the algebraic reconstruction should be consistent with Bohr’s original view, the language that speaks of the beables should obey classical logic and theorems of classical mathematics. The purpose of this paper is to rigorously realize this interpretation of Bohr’s view in the framework of quantum set theory.

Recall that the logic of observational propositions on the system \mathbf{S} is represented by the projection lattice $\mathcal{P}(\mathcal{H})$ of all projections on \mathcal{H} . In order to treat a context-dependent part of the whole observational propositions, we consider a sublogics of $\mathcal{P}(\mathcal{H})$ represented by the projection lattice $\mathcal{P}(\mathcal{M})$ of a von Neumann subalgebra \mathcal{M} of $\mathcal{L}(\mathcal{H})$. Let \mathcal{M} be a von Neumann algebra on \mathcal{H} and denote by $\mathcal{O}(\mathcal{M})$ the set of observables affiliated with \mathcal{M} . The observational propositions on the observables affiliated with \mathcal{M} are constructed from *atomic propositions* “ $X \leq_o \lambda$ ”, where $X \in \mathcal{O}(\mathcal{M})$ and $\lambda \in \mathbb{R}$, by connecting them with \wedge (conjunction), \vee (disjunction), \rightarrow (conditional), \leftrightarrow (equivalence), and \neg (negation). In what follows, we consider only the conjunction and negation as primitive symbols and the other connectives as derived symbols by the following definitions: $\varphi_1 \vee \varphi_2 = \neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \rightarrow \varphi_2 = \neg\varphi_1 \vee (\varphi_1 \wedge \varphi_2)$, $\varphi_1 \leftrightarrow \varphi_2 = (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$. Here, we note that the conditional \rightarrow is defined as the Sasaki conditional [35]. We denote by $\mathbf{L}(\mathcal{M})$ the set of observational propositions on the observables in $\mathcal{O}(\mathcal{M})$. We also write $\mathbf{L}(\mathcal{H}) = \mathbf{L}(\mathcal{L}(\mathcal{H}))$. We have $\mathbf{L}(\mathcal{M}_1) \subseteq \mathbf{L}(\mathcal{M}_2)$ if $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

Following Birkhoff and von Neumann [3], every observational proposition φ has the projection-valued truth value $[[\varphi]]_o$ determined by the following rules.

- (i) $[[X \leq_o \lambda]]_o = E^X(\lambda)$.
- (ii) $[[\varphi_1 \wedge \varphi_2]]_o = [[\varphi_1]]_o \wedge [[\varphi_2]]_o$.
- (iii) $[[\neg\varphi]]_o = [[\varphi]]_o^\perp$.

Then for every observational proposition φ we define the probability that the observational proposition φ holds in the state ψ by $\Pr\{\varphi\|\psi\} = \|[[\varphi]]_o\psi\|^2$. It is well-know that the logic of observational propositions in $\mathbf{L}(\mathcal{H})$ is non-distributive, so that it does not necessarily follow the laws of classical logic. According to Bohr’s view on “classical description” it is natural to expect that in the state ψ the language $\mathbf{L}(\mathcal{B})$ satisfies classical logic and all the mathematical theorems, if \mathcal{B} is beable for the state ψ . However, the language $\mathbf{L}(\mathcal{M})$ has only a limited power in expressing observational relations between observables, just as the propositional logic has only a limited power in expressing relations between mathematical objects. In fact, the language $\mathbf{L}(\mathcal{M})$ cannot generally express relations between observables in $\mathcal{O}(\mathcal{M})$ such as equality and order relation, so that it cannot assign the projection valued truth-value, or the probability in the state ψ , for the proposition “A and B have the same value”. In what follows we strengthen the language of observational propositions to have full power of expressing all the mathematically definable relations between observables with projection-valued truth values.

4 Quantum set theory

In this and the next sections, we shall introduce basic facts about quantum set theory. We refer the reader to Ref. [31] for detailed formulations. Let \mathcal{M} be a von Neumann algebra. The purpose of quantum set theory is to extend the universe, or a ground model, V of ZFC set theory to a sort of “generic extension” $V[\mathcal{M}]$ adding self-adjoint operators affiliated with \mathcal{M} as “generic reals” to V .

We denote by V the universe of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC). Let $\mathbf{L}(\in)$ be the language for first-order theory with equality augmented by a binary relation symbol \in , bounded quantifier symbols $\forall x \in y$, $\exists x \in y$, and no constant symbols. For any class U , the language

$\mathbf{L}(\in, U)$ is the one obtained by adding a name for each element of U . We take the symbols $\neg, \wedge, \forall x \in y$, and $\forall x$ as primitive, and the symbols $\vee, \rightarrow, \leftrightarrow, \exists x \in y$, and $\exists x$ as derived symbols by defining:

- (i) $\varphi_1 \vee \varphi_2 = \neg(\neg\varphi_1 \wedge \neg\varphi_2)$,
- (ii) $\varphi_1 \rightarrow \varphi_2 = \neg\varphi_1 \vee (\varphi_1 \wedge \varphi_2)$,
- (iii) $\varphi_1 \leftrightarrow \varphi_2 = (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$,
- (iv) $\exists x \in y \varphi(x) = \neg(\forall x \in y \neg\varphi(x))$,
- (v) $\exists x \varphi(x) = \neg(\forall x \neg\varphi(x))$.

Let \mathcal{M} be a von Neumann algebra on a Hilbert space \mathcal{H} . For each ordinal α , let

$$V_\alpha[\mathcal{M}] = \{u \mid u : \text{dom}(u) \rightarrow \mathcal{P}(\mathcal{M}) \text{ and } (\exists \beta < \alpha) \text{dom}(u) \subseteq V_\beta[\mathcal{M}]\}.^8 \quad (5)$$

The $\mathcal{P}(\mathcal{M})$ -valued universe $V[\mathcal{M}]$ is defined by

$$V[\mathcal{M}] = \bigcup_{\alpha \in \text{On}} V_\alpha[\mathcal{M}], \quad (6)$$

where On is the class of all ordinals. We shall write $V_\alpha[\mathcal{H}] = V_\alpha[\mathcal{L}(\mathcal{H})]$ and $V[\mathcal{H}] = V[\mathcal{L}(\mathcal{H})]$. For every $u \in V[\mathcal{M}]$, the *rank* of u , denoted by $\text{rank}(u)$, is defined as the least α such that $u \in V_{\alpha+1}[\mathcal{M}]$. It is easy to see that if $u \in \text{dom}(v)$ then $\text{rank}(u) < \text{rank}(v)$.

We introduce the implication operation \rightarrow and its dual conjunction operation $*$ on the lattice $\mathcal{P}(\mathcal{M})$ by $P \rightarrow Q = P^\perp \vee (P \wedge Q)$ and $P * Q = P \wedge (P^\perp \vee Q)$, or equivalently $P * Q = (P \rightarrow Q^\perp)^\perp$, for any $P, Q \in \mathcal{P}(\mathcal{M})$. The operation \rightarrow on $\mathcal{P}(\mathcal{M})$ is called the Sasaki arrow and the operation $Q \mapsto P * Q$ is called the Sasaki projection [21].

For any $u, v \in V[\mathcal{M}]$, the $\mathcal{P}(\mathcal{M})$ -valued truth values of atomic formulas $u = v$ and $u \in v$ are assigned by the following rules recursive in rank.

- (vi) $\llbracket u = v \rrbracket_{\mathcal{M}} = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow \llbracket u' \in v \rrbracket_{\mathcal{M}}) \wedge \bigwedge_{v' \in \text{dom}(v)} (v(v') \rightarrow \llbracket v' \in u \rrbracket_{\mathcal{M}})$.
- (vii) $\llbracket u \in v \rrbracket_{\mathcal{M}} = \bigvee_{v' \in \text{dom}(v)} (v(v') * \llbracket u = v' \rrbracket_{\mathcal{M}})$.

To each statement φ of $\mathbf{L}(\in, V[\mathcal{M}])$ we assign the $\mathcal{P}(\mathcal{M})$ -valued truth value $\llbracket \varphi \rrbracket_{\mathcal{M}}$ by the following rules.

- (viii) $\llbracket \neg\varphi \rrbracket_{\mathcal{M}} = \llbracket \varphi \rrbracket_{\mathcal{M}}^\perp$.
- (ix) $\llbracket \varphi_1 \wedge \varphi_2 \rrbracket_{\mathcal{M}} = \llbracket \varphi_1 \rrbracket_{\mathcal{M}} \wedge \llbracket \varphi_2 \rrbracket_{\mathcal{M}}$.
- (x) $\llbracket (\forall x \in u) \varphi(x) \rrbracket_{\mathcal{M}} = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow \llbracket \varphi(u') \rrbracket_{\mathcal{M}})$.
- (xi) $\llbracket (\forall x) \varphi(x) \rrbracket_{\mathcal{M}} = \bigwedge_{u \in V[\mathcal{M}]} \llbracket \varphi(u) \rrbracket_{\mathcal{M}}$.

We say that a statement φ of $\mathbf{L}(\in, V[\mathcal{M}])$ *holds* in $V[\mathcal{M}]$ if $\llbracket \varphi \rrbracket_{\mathcal{M}} = I$. A formula in $\mathbf{L}(\in, V[\mathcal{M}])$ is called a Δ_0 -formula iff it has no unbounded quantifiers $\forall x$ or $\exists x$. The following theorem holds [31, Theorem 4.3].

⁸We denote by $\text{dom}(f)$ the domain of a function f . By $f : D \rightarrow R$ we mean that f is a function defined on a set D with values in a set R .

Theorem 4.1 (Δ_0 -Absoluteness Principle). *For any Δ_0 -formula $\varphi(x_1, \dots, x_n)$ of $\mathbf{L}(\in)$ and $u_1, \dots, u_n \in V[\mathcal{M}]$, we have*

$$\llbracket \varphi(u_1, \dots, u_n) \rrbracket_{\mathcal{M}} = \llbracket \varphi(u_1, \dots, u_n) \rrbracket_{\mathcal{L}(\mathcal{H})}. \quad (7)$$

Henceforth, for any Δ_0 -formula $\varphi(x_1, \dots, x_n)$ and $u_1, \dots, u_n \in V[\mathcal{M}]$, we abbreviate $\llbracket \varphi(u_1, \dots, u_n) \rrbracket = \llbracket \varphi(u_1, \dots, u_n) \rrbracket_{\mathcal{M}}$, which is the common $\mathcal{P}(\mathcal{H})$ -valued truth value for $u_1, \dots, u_n \in V[\mathcal{H}]$.

The universe V can be embedded in $V[\mathcal{M}]$ by the following operation $\check{\vee} : v \mapsto \check{v}$ defined by the \in -recursion: for each $v \in V$, $\check{v} = \{\check{u} \mid u \in v\} \times \{I\}$. Note that $\check{v} \in V[\mathcal{M}]$ for any $v \in V$ and any von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$. Then we have the following [31, Theorem 4.8].

Theorem 4.2 (Δ_0 -Elementary Equivalence Principle). *For any Δ_0 -formula $\varphi(x_1, \dots, x_n)$ of $\mathbf{L}(\in)$ and $u_1, \dots, u_n \in V$, we have $\langle V, \in \rangle \models \varphi(u_1, \dots, u_n)$ if and only if $\llbracket \varphi(\check{u}_1, \dots, \check{u}_n) \rrbracket = I$.*

Thus, $V[\mathcal{M}]$ includes (a copy of) the standard universe V as a Δ_0 -elementary equivalent submodel. For further detail about the universe $V[\mathcal{M}]$ we refer the reader to [31].

5 Transfer principle

For any $u \in V[\mathcal{M}]$, we define the *support* of u , denoted by $L(u)$, by transfinite recursion on the rank of u by the relation

$$L(u) = \bigcup_{x \in \text{dom}(u)} L(x) \cup \{u(x) \mid x \in \text{dom}(u)\} \cup \{0\}. \quad (8)$$

For $\mathcal{A} \subseteq V[\mathcal{M}]$ we write $L(\mathcal{A}) = \bigcup_{u \in \mathcal{A}} L(u)$ and for $u_1, \dots, u_n \in V[\mathcal{M}]$ we write $L(u_1, \dots, u_n) = L(\{u_1, \dots, u_n\})$. Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$. The *commutant* of \mathcal{A} in $\mathcal{L}(\mathcal{H})$, denoted by \mathcal{A}' , is defined by

$$\mathcal{A}' = \{A \in \mathcal{L}(\mathcal{H}) \mid [A, B] = 0 \text{ for all } B \in \mathcal{A}\}, \quad (9)$$

and the *commutant* of \mathcal{A} in $\mathcal{P}(\mathcal{H})$, denoted by $\mathcal{A}^!$, is defined by $\mathcal{A}^! = \mathcal{A}' \cap \mathcal{P}(\mathcal{H})$.

Let $\mathcal{A} \subseteq \mathcal{P}(\mathcal{H})$. Takeuti [37] introduced the *commutator* of \mathcal{A} , denoted by $\text{com}(\mathcal{A})$, given by

$$\text{com}(\mathcal{A}) = \bigvee \{E \in \mathcal{A}' \cap \mathcal{A}^! \mid [P, Q]E = 0 \text{ for all } P, Q \in \mathcal{A}\}. \quad (10)$$

For any $P_1, \dots, P_n \in \mathcal{P}(\mathcal{H})$, we write $\text{com}(\{P_1, \dots, P_n\}) = \text{com}(P_1, \dots, P_n)$. We refer the reader to [28] for further properties of commutators.

Let $\mathcal{A} \subseteq V[\mathcal{M}]$. The *commutator* of \mathcal{A} , denoted by $\text{com}(\mathcal{A})$, is defined by

$$\text{com}(\mathcal{A}) = \text{com}(L(\mathcal{A})). \quad (11)$$

For any $u_1, \dots, u_n \in V[\mathcal{M}]$, we write $\text{com}(u_1, \dots, u_n) = \text{com}(\{u_1, \dots, u_n\})$.

We have the following transfer principle for bounded theorems of ZFC [31, Theorem 4.15].

Theorem 5.1 (Δ_0 -ZFC Transfer Principle). *For any Δ_0 -formula $\varphi(x_1, \dots, x_n)$ of $\mathbf{L}(\in)$ and $u_1, \dots, u_n \in V[\mathcal{M}]$, if $\varphi(x_1, \dots, x_n)$ is provable in ZFC, then*

$$\llbracket \varphi(u_1, \dots, u_n) \rrbracket \geq \text{com}(u_1, \dots, u_n). \quad (12)$$

6 Internal real numbers in quantum set theory

Let \mathbb{Q} be the set of rational numbers in V . We define the set of rational numbers in the model $V[\mathcal{M}]$ to be $\check{\mathbb{Q}}$. We define a real number in the model by a Dedekind cut of the rational numbers. More precisely, we identify a real number with the upper segment of a Dedekind cut whose lower segment has no end point. Therefore, the formal definition of the predicate $\mathbb{R}(x)$, “ x is a real number,” is expressed by

$$\begin{aligned} \mathbb{R}(x) \quad := \quad & \forall y \in x (y \in \check{\mathbb{Q}}) \wedge \exists y \in \check{\mathbb{Q}} (y \in x) \wedge \exists y \in \check{\mathbb{Q}} (y \notin x) \\ & \wedge \forall y \in \check{\mathbb{Q}} (y \in x \leftrightarrow \forall z \in \check{\mathbb{Q}} (y < z \rightarrow z \in x)). \end{aligned} \quad (13)$$

We define $\mathbb{R}[\mathcal{M}]$ to be the interpretation of the set \mathbb{R} of real numbers in $V[\mathcal{M}]$ as follows.

$$\mathbb{R}[\mathcal{M}] = \{u \in V[\mathcal{M}] \mid \text{dom}(u) = \text{dom}(\check{\mathbb{Q}}) \text{ and } \llbracket \mathbb{R}(u) \rrbracket = I\}. \quad (14)$$

For any $u \in \mathbb{R}[\mathcal{M}]$ and $\lambda \in \mathbb{R}$, we define $E^u(\lambda)$ by

$$E^u(\lambda) = \bigwedge_{\lambda < r \in \mathbb{Q}} u(\check{r}). \quad (15)$$

Then it can be shown that $\{E^u(\lambda)\}_{\lambda \in \mathbb{R}}$ is a resolution of identity in $\mathcal{P}(\mathcal{M})$ and hence by the spectral theorem there is an observable $\hat{u} \in \mathcal{O}(\mathcal{M})$ uniquely satisfying $\hat{u} = \int_{\mathbb{R}} \lambda dE^u(\lambda)$. On the other hand, let $A \in \mathcal{O}(\mathcal{M})$. We define $\tilde{A} \in V[\mathcal{M}]$ by

$$\tilde{A} = \{(\check{r}, E^A(r)) \mid r \in \mathbb{Q}\}. \quad (16)$$

Then $\text{dom}(\tilde{A}) = \text{dom}(\check{\mathbb{Q}})$ and $\tilde{A}(\check{r}) = E^A(r)$ for all $r \in \mathbb{Q}$. It is easy to see that $\tilde{A} \in \mathbb{R}[\mathcal{M}]$ and we have $(\hat{u})^\sim = u$ for all $u \in \mathbb{R}[\mathcal{M}]$ and $(\tilde{A})^\wedge = A$ for all $A \in \mathcal{O}(\mathcal{M})$. Therefore, the correspondence between $\mathbb{R}[\mathcal{M}]$ and $\mathcal{O}(\mathcal{M})$ is bijective. We call the above correspondence the *Takeuti correspondence*. Now, we have the following.

Theorem 6.1. *The relations*

$$(i) \quad E^A(\lambda) = \bigwedge_{\lambda < r \in \mathbb{Q}} u(\check{r}) \text{ for all } \lambda \in \mathbb{Q},$$

$$(ii) \quad u(\check{r}) = E^A(r) \text{ for all } r \in \mathbb{Q},$$

where $u = \tilde{A} \in \mathbb{R}[\mathcal{M}]$ and $A = \hat{u} \in \mathcal{O}(\mathcal{M})$, sets up a bijective correspondence between $u \in \mathbb{R}[\mathcal{M}]$ and $A \in \mathcal{O}(\mathcal{M})$.

For any $r \in \mathbb{R}$, we shall write $\tilde{r} = (r1)^\sim$, where $r1$ is the scalar operator on \mathcal{H} . Then we have $\text{dom}(\tilde{r}) = \text{dom}(\check{\mathbb{Q}})$ and $\tilde{r}(\check{r}) = \llbracket \check{r} \leq \check{r} \rrbracket$, so that we have $L(\tilde{r}) = \{0, 1\}$. The order relation for $u, v \in \mathbb{R}[\mathcal{M}]$ is naturally defined by

$$u \leq v := (\forall x \in \check{\mathbb{Q}}) [x \in v \rightarrow x \in u]. \quad (17)$$

Recall that a formula $\varphi(x_1, \dots, x_n) \in \mathbf{L}(\in, V[\mathcal{M}])$ is called a Δ_0 -formula iff it contains no unbounded quantifiers $\forall x$ nor $\exists x$. In this paper, we focus on the sublanguage $\mathbf{L}_0(\in, V[\mathcal{M}])$ consisting of Δ_0 -formulas in $\mathbf{L}(\in, V[\mathcal{M}])$. Then, for every statement $\varphi \in \mathbf{L}_0(\in, V[\mathcal{M}])$ we have the $\mathcal{P}(\mathcal{M})$ -valued truth value $\llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket_{\mathcal{P}(\mathcal{M})}$, and for every observable $X \in \mathcal{O}(\mathcal{M})$ we have a real number $\tilde{X} \in \mathbb{R}[\mathcal{M}]$ in $V[\mathcal{M}]$. Thus, there is an embedding of every observational proposition φ in the language $\mathbf{L}(\mathcal{M})$ into a statement $\tilde{\varphi}$ in the language $\mathbf{L}_0(\in, V[\mathcal{M}])$ defined by the following rules for any $X \in \mathcal{O}(\mathcal{M})$ and $x \in \mathbb{R}$, and observational propositions $\varphi, \varphi_1, \varphi_2$:

- (Q1) $\widetilde{X \leq_o x} := \tilde{X} \leq \tilde{x}$.
(Q2) $\widetilde{\neg \varphi} := \neg \tilde{\varphi}$.
(Q3) $\widetilde{\varphi_1 \wedge \varphi_2} := \tilde{\varphi}_1 \wedge \tilde{\varphi}_2$.

Then, it is easy to see that the relation

$$[[\tilde{\varphi}]] = [[\varphi]]_o \quad (18)$$

holds for any observational proposition φ .

Thus, all the observational propositions are embedded in a set of statements in $\mathbf{L}_0(\in, V[\mathcal{M}])$ preserving their projection-valued truth values. Quantum set theory provides a language strong enough to express all the possible mathematical relations among observables. For every mathematical relation $R(x_1, \dots, x_n)$ definable in ZFC set theory, we can assign the quantum logical truth value $[[R(\tilde{X}_1, \dots, \tilde{X}_n)]] \in \mathcal{P}(\mathcal{M})$ of the relation $R(\tilde{X}_1, \dots, \tilde{X}_n)$ for any observables $X_1, \dots, X_n \in \mathcal{O}(\mathcal{H})$. For example, we can generally define the projection valued truth values for the order relation and the equality relation for observables so that the following relation hold:

$$[[\tilde{X} \leq \tilde{Y}]] = [[(\forall x \in \tilde{\mathcal{Q}}) [\tilde{Y} \leq x \rightarrow \tilde{X} \leq x]]], \quad (19)$$

$$[[\tilde{X} = \tilde{Y}]] = [[\tilde{X} \leq \tilde{Y} \wedge \tilde{Y} \leq \tilde{X}]]. \quad (20)$$

7 Beable subuniverses

Let $\psi \in \mathcal{H}$ be a state. We say that a statement $\varphi(u_1, \dots, u_n) \in \mathbf{L}_0(\in, V[\mathcal{M}])$ holds in ψ , and write $\psi \Vdash \varphi(u_1, \dots, u_n)$, iff $\Pr\{\varphi(u_1, \dots, u_n) \mid \psi\} = I$. For any von Neumann algebra $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$, the subuniverse $V[\mathcal{M}] \subseteq V[\mathcal{H}]$ is said to be *ZFC-satisfiable in ψ* iff

$$\psi \Vdash \phi(u_1, \dots, u_n) \quad (21)$$

for any $u_1, \dots, u_n \in V[\mathcal{M}]$ and any formula $\phi(x_1, \dots, x_n) \in \mathbf{L}_0(\in)$ provable in ZFC.

The following theorem holds.

Theorem 7.1. *Let \mathcal{M} be a von Neumann algebra on \mathcal{H} . Then the following conditions are all equivalent.*

- (i) $V[\mathcal{M}]$ is ZFC-satisfiable in ψ .
- (ii) \mathcal{M} is beable for ψ .
- (iii) $[X, Y]\psi = 0$ for any $X, Y \in \mathcal{M}$.
- (iv) $R(\psi) \leq \text{com}(u_1, \dots, u_n)$ for any $u_1, \dots, u_n \in V(\mathcal{M})$.

8 Context-definable beable subuniverses

Let (ψ, A) be a measurement context. For any unitary operator U on \mathcal{H} , we define $\alpha_U : V[\mathcal{H}] \rightarrow V[\mathcal{H}]$ by transfinite recursion on the rank of $u \in V[\mathcal{H}]$ as

$$\alpha_U(u) = \{\langle \alpha_U(x), U^*u(x)U \rangle \mid x \in \text{dom}(u)\}. \quad (22)$$

A subclass $\mathcal{U} \subseteq V[\mathcal{M}]$ is called *definable* by (ψ, A) iff

- (i) $\tilde{A} \in \mathcal{U}$,
- (ii) $\alpha_U(\mathcal{U}) \subseteq \mathcal{U}$ for any unitary operator U on \mathcal{H} satisfying $[A, U] = 0$ and $U\psi = \psi$.

A subuniverse $V[\mathcal{M}] \subseteq V[\mathcal{H}]$ is called a *maximally definable ZFC-satisfiable subuniverse* for the measurement context (ψ, A) iff $V[\mathcal{M}]$ is a ZFC-satisfiable subuniverse in ψ definable by (ψ, A) and there is no ZFC-satisfiable subuniverse $V[\mathcal{M}_0]$ definable by (ψ, A) that properly includes $V[\mathcal{M}]$. Then, we have the following.

Theorem 8.1. (i) A subuniverse $V[\mathcal{M}]$ is a ZFC-satisfiable subuniverse of $V[\mathcal{H}]$ definable by (ψ, A) if and only if \mathcal{M} is a beable subalgebra of $\mathcal{L}(\mathcal{H})$ definable by (ψ, A) .

(ii) There uniquely exists a maximally definable ZFC-satisfiable subuniverse $V[\mathcal{M}] \subseteq V[\mathcal{H}]$ for any measurement context (ψ, A) . In this case, \mathcal{M} is of the form $\mathcal{M} = \mathcal{B}(\psi, A)$, i.e.,

$$\mathcal{M} = W^*(A)P \oplus \mathcal{L}(P^\perp \mathcal{H}), \quad (23)$$

where P is the projection from \mathcal{H} onto the cyclic subspace $\mathcal{C}(\psi, A)$ of \mathcal{H} generated by ψ and A .

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